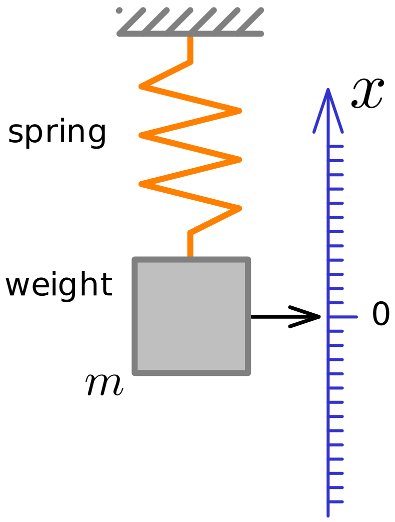
2. Oscillators and waves

Notes to the theory, Review of physics 2, Martin Žáček, zacekm@fel.cvut.cz

2.1. Linear harmonic oscillator

Mechanical model:



Equation of motion:

$F = ma$ Newton's equation of motion (1)

$F = -kx$ spring reaction force (2)

k rigidity, $[k] = \mathrm{N/m}$

x displacement

Rigidity is a general term for any oscillating system, in this mechanical model it is the coefficient of proportionality between tension and the extension of the spring. Reaction force i negative because it direction is opposite to the displacement.

Substituting (1) into (2) we get

$$ma + kx = 0 \quad \rightarrow \quad a + \frac{k}{m}x = 0 \quad \rightarrow \quad \boxed{a + \omega^2 x = 0} \quad (3),$$

equation of the linear harmonic oscillator

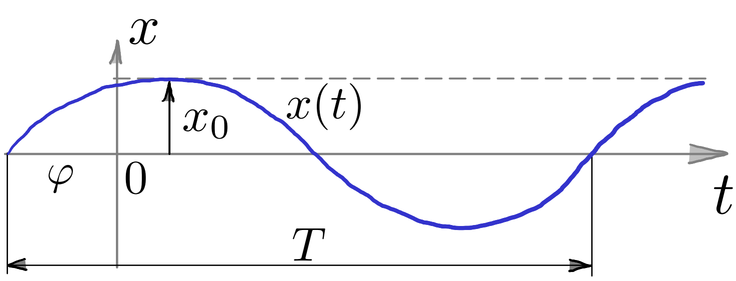
where $\boxed{\omega = \sqrt{\frac{k}{m}}}$ (4) is the circular frequency, $[\omega] = 1/\mathrm{s}$.

$f = \frac{\omega}{2\pi}$ frequency, $[f] = 1/\mathrm{s}$

$T = \frac{1}{f} = \frac{2\pi}{\omega}$ period, $[T] = \mathrm{s}$

The equation (3) has the following solution
(derivable by methods of solving differential equations):

$$x(t) = x_0 \cos(\omega t + \varphi) \quad (5)$$



x_0 the amplitude

φ initial phase angle

Two coeficients x_0 and φ are optional, in fact they play role as integration constant, ω is known parameter given by formula (4), t is the variable.

Velocity and acceleration:

can be calculated as time derivatives:

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}\,, \qquad a = \frac{\mathrm{d}v}{\mathrm{d}t} = \dot{v} = \ddot{x}$$

aplied to the oscillations (5) we get

$$v(t) = -x_0\omega \sin(\omega t + \varphi) = v_0 \cos(\omega t + \varphi_2)$$
$$a(t) = -x_0\omega^2 \cos(\omega t + \varphi) = a_0 \cos(\omega t + \varphi_3) \quad (6)$$

where the amplitudes of velocity and acceleration are

$$v_0 = x_0\omega$$
$$a_0 = x_0\omega^2$$

velocity and acceleration amplitudes
for a linear harmonic oscillator (7)

The sign and sine instead of the cosine actually change the initial angle, and the expressions can be converted to a positive sign cosine according to the formulas

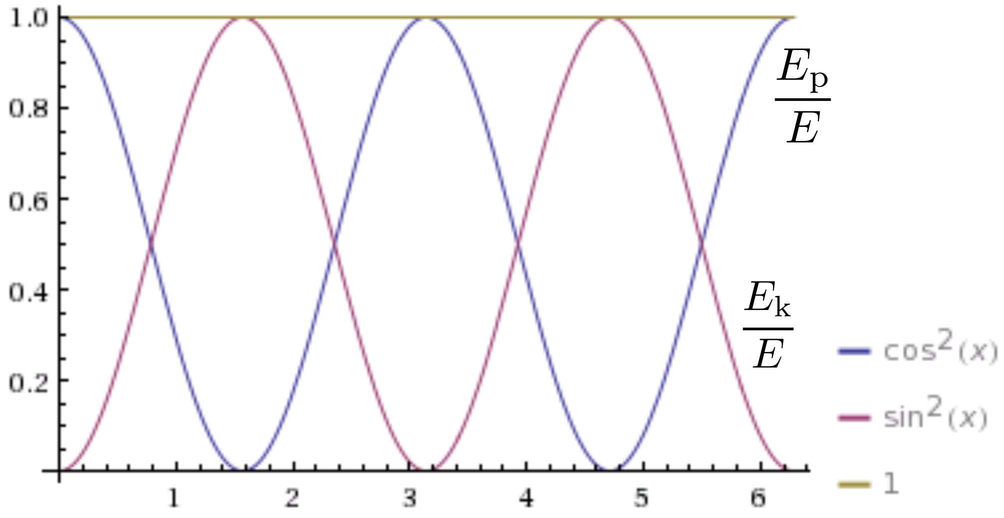
$$\underbrace{\sin(\alpha) = \cos(\alpha - \frac{\pi}{2}),}_{\text{shift by a quarter of the period}} \quad \underbrace{\sin(-\alpha) = -\sin(\alpha),}_{\text{sine is an odd function}} \quad \underbrace{\cos(-\alpha) = \cos(\alpha).}_{\text{cosine is an even function}}$$

It can also be seen now that in formulation (5) it is possible to choose a sine instead of a cosine, with a suitable change of the initial phase.

Energy of the linear harmonic oscilator:

is given as sum of kinetic and potential energy:

$$E = E_\mathrm{k} + E_\mathrm{p} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \underset{k=m\omega^2}{\underset{\uparrow}{=}} \frac{1}{2}m(v^2 + \omega^2 x^2) \underset{(6)}{\underset{\uparrow}{=}} \frac{1}{2}m\omega^2 x_0^2 (\underbrace{\sin^2(\omega t + \varphi) + \cos^2(\omega t + \varphi)}_{1})$$



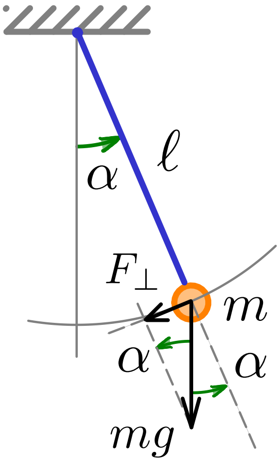
so the final formula is:

$$E = \frac{1}{2}m\omega^2 x_0^2 = \mathrm{const}$$

total energy of an linear harmonic oscilator

While the kinetic and potential components of the energy are changed during the time, their sum remains constant. But it is expected because the oscilator represents autonomous energetically isolated system.

2.2. Pendulum



Reaction force: $F_\perp = -mg\sin\alpha$ (see picture).

For small deviation α is $\sin\alpha \approx \alpha$.
Setting to the (1) we get (with angular acceleration $\varepsilon = \frac{a}{\ell}$)

$$-mg\alpha = ma \quad \rightarrow \quad -g\alpha = \ell\varepsilon \quad \rightarrow \quad \underbrace{\varepsilon + \frac{g}{\ell}\alpha = 0}$$

equation of the motion for
pendulum fom small deviation

As the equation is formally the same as (3), the solution is formally the same

$$\alpha(t) = \alpha_0 \cos(\omega t + \varphi)\,, \quad \text{where} \quad \boxed{\omega = \sqrt{\frac{g}{\ell}}}\,.$$

circular frequency of pendulum
for case of small amplitude

Note that the frequency of the pendulum does not depend on the weight.