# Notes on the relativistic movement of runaway electrons in parallel electric and magnetic fields

Vojtěch Adalbert Delong,[1,2] Radek Beňo,[1] David Břeň,[3] and Petr Kulhánek[1,a)]
[1)] *Czech Technical University, Faculty of Electrical Engineering, Technická 2, 162 00 Prague 6, Czech Republic*
[2)] *Institute of Physics, Czech Academy of Sciences, Cukrovarnická 10, 162 00 Prague 6, Czech Republic*
[3)] *Czech Technical University, Faculty of Nuclear Sciences and Physical Engineering, Břehová 7, 115 19 Prague 1, Czech Republic*

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Runaway electrons are a potential threat in many plasma devices. At high velocities, the plasma acceleration is not further offset by collisions in the plasma, as in the ohmic regime. The particles obtain relativistic velocity and considerable energy. A typical configuration includes parallel electric and magnetic fields, in which there are no drifts, and the movement of the charged particles is a combination of gyration motion with the acceleration in an electric field. It follows from the Lorentz equation of motion that the transverse velocity component (perpendicular to the fields) will be interconnected with the longitudinal component via the Lorentz factor. The increasing longitudinal velocity will therefore ultimately reduce the magnitude of the transverse velocity component, thereby decreasing the gyrofrequency. The corresponding change in Larmor radius will be offset by the increase in the particle mass and the Larmor radius of gyration therefore remains unchanged. We derive analytical relations for the temporal and spatial dependences of frequency, and longitudinal and transverse components of the velocity.

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## I. INTRODUCTION

The interaction of the charged particle beam (for definiteness we consider electrons) with the Maxwellian plasma target in the presence of an electric field leads under normal conditions to a balance between collision and accelerating processes. If the speed is slightly increased above the equilibrium rate, the collision frequency will be higher, and the velocity of the electron returns to its original value. Conversely, when the speed of the electron is reduced randomly, the collision frequency decreases, acceleration prevails, and the velocity of the electron will again return to its original value. This equilibrium is called the ohmic regime. It can be derived from the Fokker-Planck equation, that there are two possible ways to disrupt the ohmic regime: 1) increasing the electric field above the critical limit called Dreicer field; 2) increasing the speed above the critical value.[1–3] In both cases an acceleration which is not sufficiently offset by collision processes prevails and the electron falls into the so-called Runaway mode.

Runaway Electrons (RE) originate in both space and laboratory plasmas.[4–6] They can cause considerable problems in both cases. One of the examples in nearby plasma are Van Allen radiation belts, in which the RE population is a potential threat for the devices as well as for the astronauts. Runaway electrons in laboratory plasmas can have a negative impact on various technologies; recently they have become a worrisome threat for larger tokamaks including the recently built ITER tokamak.[7,8]

Runaway electrons can gain considerable energy, which is subsequently lost through many channels, such as bremsstrahlung radiation,[9] synchrotron radiation,[10] creation of electron-positron pairs,[11,12] and collisions with surrounding plasma, which may lead to an avalanche effect.[13] Today's physicists have limited knowledge about the formation of runaway electrons and about their future fate. At present there is no known consistent equation of motion that would describe the motion of charged particles, including its reaction to the own radiation field. The Lorenz-Dirac equation (including its low energetic limit, the Abraham-Lorenz equation) has a number of problems.[14] The presence of the third time derivative of positions leads to unclear initial conditions, some solutions grow exponentially even if no force is present, and some solutions are not causal in the sense that at any given time they depend on the value of force in the future. These problems are partially solved by expansion in a series that artificially suppresses the unwanted solutions.[15,16]

We will concentrate on the movement of charged particle in runaway regime (with negligible collisions), in parallel electric and magnetic fields, described by the Lorentz equation of motion. Such configuration does not lead to drifts and is therefore ideal for investigation of motion at speeds comparable to the speed of light (of course, knowing that this description does not include the reaction of the particle to its own radiation field). We will show that useful analytic expressions for Larmor radius,

[a)]Electronic mail: kulhanek@fel.cvut.cz

gyrofrequency, and the longitudinal and transverse velocity components can be derived from the Lorentz equation of motion.

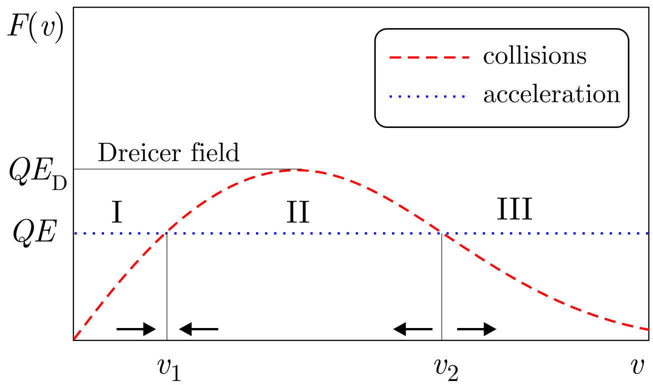


FIG. 1. Forces acting on the electron. The dashed line represents the dependence of the collision force on the particle speed, and the dotted line represents the acceleration force. If the field is lower than corresponds to the maximum of the dashed curve (Dreicer field), there are two possibilities: the electron can either be in the ohmic regime (regions I and II), or in the Runaway mode (region III).

## II. LARMOR (CYCLOTRON) RADIUS

Let us use Cartesian coordinates, in which both fields point along the $z$ axis, i.e.

$$\mathbf{B} = (0, 0, B)\,, \tag{1}$$
$$\mathbf{E} = (0, 0, E)\,. \tag{2}$$

The Lorentz equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma m_0 \mathbf{v}\right) = Q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{3}$$

leads to a system of equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma\, v_x\right) &= \omega_{\mathrm{c}} v_y\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma\, v_y\right) &= -\omega_{\mathrm{c}} v_x\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma\, v_z\right) &= \frac{QE}{m_0}\,. \end{aligned} \tag{4}$$

The first two equations are linked to the third by the Lorenz factor

$$\gamma = \frac{1}{\sqrt{1 - (v_x^2 + v_y^2 + v_z^2)/c^2}} \tag{5}$$

Let us suppose that $\gamma$ is any at least once continuously differentiable function of time with domain $(0; c)$ and with range $(1; \infty)$. We will consider for the moment only the first two equations, thus we will deal with the transverse ($\perp$) projection of the motion to the $(x, y)$ plane perpendicular on both fields. Let us introduce a vector

$$\mathbf{K}_\perp \equiv \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma\, \mathbf{v}_\perp\right) = \left(\omega_{\mathrm{c}} v_y, -\omega_{\mathrm{c}} v_x\right). \tag{6}$$

Differentiating yields

$$\mathbf{K}_\perp \equiv \dot{\gamma}\mathbf{v}_\perp + \gamma \mathbf{a}_\perp \tag{7}$$

from where we express the perpendicular acceleration (in transverse plane)

$$\mathbf{a}_\perp = \frac{\mathbf{K}_\perp}{\gamma} - \frac{\dot{\gamma}}{\gamma}\mathbf{v}_\perp \tag{8}$$

which is a superposition of two vectors

$$\mathbf{a}_1 = -\frac{\dot{\gamma}}{\gamma}\mathbf{v}_\perp\,, \tag{9}$$
$$\mathbf{a}_2 = \frac{\mathbf{K}_\perp}{\gamma}\,. \tag{10}$$

The first vector points in the direction of the projection of the trajectory into the transverse plane (is collinear to $\mathbf{v}_\perp$), and the second vector is perpendicular to the first because

$$\mathbf{a}_1 \cdot \mathbf{a}_2 \propto \mathbf{v}_\perp \cdot \mathbf{K}_\perp = v_x \cdot \omega_c v_y - v_y \cdot \omega_c v_x = 0\,. \tag{11}$$

The second vector is thus perpendicular to the trajectory projection into the transverse plane and represents the centripetal acceleration. With its help we can calculate the osculating circle radius of curvature

$$R = \frac{v_\perp^2}{|\mathbf{a}_2|} = \frac{v_\perp^2}{\frac{|\omega_{\mathrm{c}}|}{\gamma}\sqrt{v_x^2 + v_y^2}} = \frac{\gamma\, v_\perp}{|\omega_{\mathrm{c}}|}\,. \tag{12}$$

We now show that the osculating circle radius of curvature of the trajectory projection into the transverse plane is constant and therefore the projection is, in fact, identical to circular motion. It is sufficient to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma v_\perp\right) = 0\,. \tag{13}$$

We define a vector

$$\mathbf{u}_\perp \equiv \gamma \mathbf{v}_\perp. \tag{14}$$

Proof of the aforementioned statement is now relatively straightforward:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma v_\perp\right) = \frac{\mathrm{d}}{\mathrm{d}t}u_\perp = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{u_x^2 + u_y^2} \;=\; \frac{\mathbf{u}_\perp \cdot \dot{\mathbf{u}}_\perp}{u_\perp}.$$

It now follows clearly from Eq. (6) that

$$\frac{\mathrm{d}\mathbf{u}_\perp}{\mathrm{d}t} = \mathbf{K}_\perp\,, \tag{15}$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma v_\perp) \quad \propto \quad \mathbf{u}_\perp \cdot \mathbf{K}_\perp = (\gamma v_x, \gamma v_y)\cdot(\omega_\mathrm{c} v_y, -\omega_\mathrm{c} v_x) = 0\,. \tag{16}$$

This shows that the trajectory projection is indeed a circular motion with the Larmor radius given by the expression (12)

$$R_\mathrm{L} = \frac{\gamma\, v_\perp}{|\omega_\mathrm{c}|}\,. \tag{17}$$

We can re-write the expression using individual components of the velocity ($v_r = 0$, therefore $v_\perp = v_\varphi$)

$$R_\mathrm{L} = \frac{v_\varphi}{|\omega_\mathrm{c}|\sqrt{1-(v_\varphi^2+v_z^2)/c^2}}\,. \tag{18}$$

Increase of the parallel velocity component caused by the electric field acceleration is compensated by the decrease of the azimuthal velocity component. Because of this the Larmor radius remains constant. For future reference, let us write down the Larmor gyroradius (17) via the initial conditions:

$$R_\mathrm{L} = \frac{u_0}{|\omega_\mathrm{c}|}\ ;\ , \tag{19}$$

$$u_0 \equiv \frac{v_{\varphi 0}}{\sqrt{1-(v_{\varphi 0}^2+v_{z0}^2)/c^2}}\,. \tag{20}$$

## III. DEPENDENCE OF THE VELOCITY COMPONENTS ON TIME

From the last, thus far neglected, part of the equation of motion (4), it follows that

$$\gamma v_z = \frac{QE}{m_0}(t-t_0)\ , \tag{21}$$

where $t_0$ is the time coordinate in past, when the parallel velocity component is zero. It holds for a trajectory with fixed Larmor radius that

$$\gamma = \frac{1}{\sqrt{1-(v_\varphi^2+v_z^2)/c^2}}\,. \tag{22}$$

From the system of equations (18), (21), and (22) we can, after some straightforward manipulations, calculate the time dependencies

$$\begin{aligned} v_\varphi(\tau) &= \frac{u_0}{\sqrt{1+u_0^2/c^2}}\frac{1}{\sqrt{1+\tau^2}}\,,\\ v_z(\tau) &= \frac{c}{\sqrt{1+\tau^{-2}}}\,,\\ \gamma(\tau) &= \sqrt{1+u_0^2/c^2}\ \sqrt{1+\tau^2}\,. \end{aligned} \tag{23}$$

where we expressed the Larmor radius using Eq. (19) and with $\tau$ denoting the dimensionless time interval calculated from the point corresponding to $t_0$:

$$\tau \equiv \frac{t-t_0}{\frac{m_0 c}{QE}\sqrt{1+u_0^2/c^2}}\,. \tag{24}$$

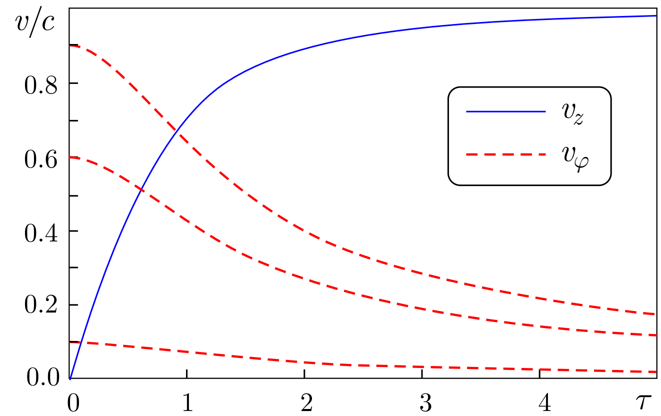


FIG. 2. Parallel velocity component increases with time and in the limit approaches the speed of light. Transverse (angular) velocity component conversely decreases (different curves correspond to different initial conditions). Radial component is zero. Dimensionless time $\tau$ is defined using formula (24).

Decrease of the angular velocity component is connected to the decrease of the gyrofrequency according to the formula

$$\omega(\tau) = \dot\varphi = \frac{v_\varphi}{R_\mathrm{L}} = \frac{1}{\sqrt{1+u_0^2/c^2}}\frac{\omega_\mathrm{c}}{\sqrt{1+\tau^2}}\,. \tag{25}$$

## IV. DEPENDENCE OF THE VELOCITY COMPONENTS ON SPACE

Let us assume the Lagrangian function of the charged particle motion in the form

$$L = -m_0c^2\sqrt{1-\frac{\mathbf{v}^2}{c^2}} - Q\phi + Q\,\mathbf{A}\cdot\mathbf{v}\,. \tag{26}$$

In the Cartesian coordinate system the corresponding potentials will be

$$\phi = -E\,z\,, \tag{27}$$

$$\mathbf{A} = \frac{1}{2}\,\mathbf{r}\times\mathbf{B} = \frac{1}{2}(yB, -xB, 0)\,. \tag{28}$$

In cylindrical coordinates the vector potential will have only one nonzero component

$$A_\varphi = \frac{1}{2}rB \tag{29}$$

and the Lagrangian function will take the form

$$L = -m_0c^2\sqrt{1-\frac{\dot r^2 + r^2\dot\varphi^2+\dot z^2}{c^2}} + QEz + \frac{1}{2}Qr^2B\,\dot\varphi\,. \tag{30}$$

The Lagrangian function does not explicitly depend on time nor on the azimuthal angle, and that is why the energy and the angular momentum given by formulas

$$W = \frac{\partial L}{\partial \dot q_k}\dot q_k = m_0\gamma c^2 - QEz\,, \tag{31}$$

$$b = \frac{\partial L}{\partial\dot\varphi} = m_0\gamma r^2\dot\varphi + \frac{1}{2}\,Q\,r^2B\,, \tag{32}$$

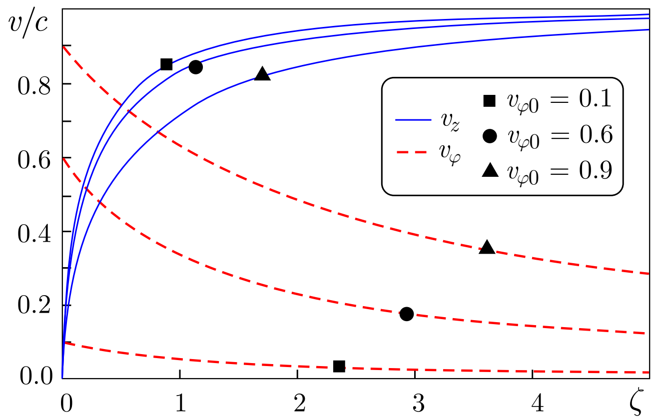


FIG. 3. Longitudinal and azimuthal velocity space dependence for different initial conditions of the azimuthal velocity component. An increase of the longitudinal velocity component is accompanied with the decrease of the azimuthal velocity component. Dimensionless variable $\varsigma$ represent space variable measured along the field and is defined using formula (35).

will be conserved. The Lorentz factor $\gamma$ is defined as

$$\gamma = \frac{1}{\sqrt{1-(\dot{r}^2+r^2\dot{\varphi}^2+\dot{z}^2)/c^2}}\,. \tag{33}$$

From the set of equations (31), (32) and (33) can be calculated (considering initial conditions and constant Larmor radius)

$$\begin{aligned} v_\varphi(\varsigma) &= \frac{u_0}{\gamma(\varsigma)}\,,\\ v_z(\varsigma) &= c\sqrt{1-\frac{1+u_0^2/c^2}{\gamma^2(\varsigma)}}\,,\\ \gamma(\varsigma) &= \gamma_0+\varsigma\,, \end{aligned} \tag{34}$$

where the dimensionless spatial coordinate in the field direction was denoted as

$$\varsigma \equiv \frac{QEz}{m_0c^2}\,. \tag{35}$$

Angular gyrofrequency decreases with longitudinal coordinate according to

$$\omega(\varsigma) = \dot{\varphi} = \frac{v_\varphi}{R_\mathrm{L}} = \frac{\omega_\mathrm{c}}{\gamma(\varsigma)}\,. \tag{36}$$

## V. CONCLUSION

In the case of parallel electric and magnetic fields it was shown from the Lorentz equation of motion that the acceleration in the direction of an electric field will lead to the decrease of azimuthal velocity and gyrofrequency of the charged particle. It was proved that the corresponding change in Larmor radius will be offset by the increase in the particle mass and the Larmor radius of gyration therefore remains unchanged even if the velocity is approaching the speed of light. If particle velocity increases in the direction of the electric field, both particle frequency and gyrofrequency decrease in the limit to zero. For velocity components and gyrofrequency analytical space and time dependencies were found. These solutions can be useful for further analysis of unsolved runaway electron problems.

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