Variational formulation of a kinetic-MHD model for relativistic runaway electrons

Alain J. Brizard
Saint Michael's College (Vermont, USA)

Fourth Runaway Electron Meeting
Pertuis (France) June 6-8, 2016

Motivation for our Work

- Investigate the interaction of a (time-dependent) population of energetic (relativistic) runaway electrons (RE) with bulk plasma (MHD) dynamics

  ◦ Can ideal MHD modes (e.g., Alfven) be driven unstable by the RE population (e.g., electron fishbone)?

  ◦ Can the RE population be modified by turbulent transport (e.g., magnetic turbulence or RF-driven quasilinear transport)?

- Kinetic-MHD model: kinetic RE coupled with bulk MHD

  ◦ Current-coupling ($\delta \mathbf{J} \times \mathbf{B}_0$) versus Pressure-coupling ($\nabla \cdot \delta \mathbf{\Pi}$)

  ◦ Variational approach is needed ($\Rightarrow$ exact conservation laws)

Outline of the Talk

- Variational formulation?
- Comments on relativistic guiding-center orderings
- Variational formulation of perturbed Vlasov-Maxwell equations
- Particle and reduced kinetic-MHD models
  - Current-coupled kinetic-MHD models
  - Pressure-coupled kinetic-MHD models
- Summary and Research outlook

Advantages of a Variational Formulation

- Self-consistent dissipationless dynamical equations have Euler-Lagrange and/or Euler-Poincaré formulations
- Noether method yields all exact dynamical conservation laws: energy-momentum, angular momentum, and wave action.
- Approximation schemes can be implemented in the variational principle itself ("perturbation-ready").
- Even reduced self-consistent dynamical equations possess exact conservation laws (e.g., gyrokinetics).
- Modular physics approach (e.g., hybrid kinetic-fluid models).
- Relevant only for dissipationless (Vlasov) dynamics

Relativistic Runaway Electron Orderings

- Maximum runaway electron gyroradius (at $p_{\|} = 0$)

$$(\rho_{\perp\mathrm{e}})_{\max} \;=\; \frac{m_{\mathrm{e}} c^2 \,(\gamma^2 - 1)^{\frac{1}{2}}}{e\; B} \;\equiv\; \rho_{\mathrm{e}} \;\simeq\; 17\,\mathrm{m}\,\left(\frac{\gamma}{B(\mathrm{G})}\right) \;=\; \frac{c}{v_{the}}\,\rho_{the}$$

  - Standard tokamak case ($B = 5\,\mathrm{T}$) for 10-100 MeV RE

$$\gamma \simeq 20 - 200 \;\;\rightarrow\;\; (\rho_{\perp\mathrm{e}})_{\max} \;\simeq\; 0.7 - 7.0\;\mathrm{cm} \;<\; L_{\mathrm{B}}$$

- Parallel guiding-center momentum ordering

$$B_{\|}^{*} \;=\; B\left(1 \;+\; \frac{p_{\|}\,c}{q\,B}\,\widehat{\mathrm{b}}\cdot\nabla\times\widehat{\mathrm{b}}\right) \;\simeq\; B\left(1 \;-\; \rho_{\mathrm{e}}\,\widehat{\mathrm{b}}\cdot\nabla\times\widehat{\mathrm{b}}\right)$$

  - Relativistic guiding-center ordering

$$\gamma\,\epsilon_{\mathrm{B}} \;\ll\; 1 \;\;\rightarrow\;\; \gamma \;\ll\; \epsilon_{\mathrm{B}}^{-1} \;\simeq\; 10^3$$

Perturbation Analysis of Vlasov-Maxwell Equations

- Wave-wave Interactions versus Dynamical Reduction

  ◦ Lowest order: wave-particle interactions (linearized equations)

  ◦ Lowest order: Guiding-center dynamical reduction (drift-kinetic)

- Wave-wave Interactions: Wave action!

  ◦ Two-wave interactions at second order: Mode Coupling

  ◦ Three-wave interactions at third order: Manley-Rowe relations

- Particle Orbit Perturbation Analysis: Lie-transform Approach

Perturbed fields $\leftrightarrow$ Perturbed particle orbits

Geometric Approach to Perturbed Particle Orbit Analysis

- Canonical phase-space transformations generated by scalar field $h \equiv$ Dynamical Hamiltonian

◦ Dynamical Hamiltonian flow generated by h:

$$\frac{\partial z^{\alpha}}{\partial t} \equiv \{z^{\alpha},\ h\}$$

- Canonical phase-space transformations generated by scalar field $S \equiv$ Perturbation Hamiltonian

◦ Perturbation Hamiltonian flow generated by S:

$$\frac{\partial z^{\alpha}}{\partial \epsilon} \equiv \{z^{\alpha},\ S\}$$

- Commuting Hamiltonian Flows (Lie-transform equation)

$$\left[\frac{d}{dt},\ \frac{d}{d\epsilon}\right] f(\mathbf{z}; t, \epsilon) \equiv 0 \Rightarrow \frac{\partial S}{\partial t} - \frac{\partial h}{\partial \epsilon} + \left\{S,\ h\right\} \equiv 0$$

Lie-transform Perturbation Theory

- Perturbation expansion: Reference $(f_0, h_0) \rightarrow$ Perturbed $(f, h; S)$

$$(f,\ h) \equiv \sum_{n=0}^{\infty} \epsilon^n \, (f_n,\ h_n) \ \text{ and } \ S \ \equiv \ \sum_{n=1}^{\infty} n\, \epsilon^{n-1}\, S_n$$

  - Lie-transform perturbation equations $(d_0/dt = \partial/\partial t + \{\,\cdot\,,\ h_0\})$

$$\frac{d_0 S_1}{dt} \ \equiv \ \frac{\partial S_1}{\partial t} + \{S_1,\ h_0\} \ = \ h_1$$

$$\frac{d_0 S_2}{dt} \ \equiv \ \frac{\partial S_2}{\partial t} + \{S_2,\ h_0\} \ = \ h_2 \ - \ \frac{1}{2}\left\{S_1,\ h_1\right\}$$

- Vlasov perturbation $(\int \delta f \ d^6 z \equiv 0)$

$$\delta f \ \equiv \ \frac{\partial f}{\partial \epsilon} \ = \ - \ \frac{\partial z^\alpha}{\partial \epsilon}\,\frac{\partial f}{\partial z^\alpha} \ \equiv \ - \ \left\{f,\ S\right\}$$

Perturbative Vlasov-Maxwell Action Functional

- Perturbation action functional $\psi_a \equiv (f, \Phi, \mathbf{A}; S, \partial_\sigma \Phi, \partial_\sigma \mathbf{A})$

$$
\begin{aligned}
\mathcal{A}_\epsilon \;=\; & \int_0^\epsilon d\sigma \int dt \left[\int d^6z \, f \left(\frac{\partial S}{\partial t} - \frac{\partial h}{\partial \sigma} + \left\{ S, \, h \right\} \right) \right] \\
& + \int_0^\epsilon d\sigma \int dt \left[\int \frac{d^3r}{4\pi} \left(\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial \sigma} - \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial \sigma} \right) \right]
\end{aligned}
$$

- Perturbation parameter σ is integrated from

Reference state $(\sigma = 0)$ $\rightarrow$ Perturbed state $(\sigma = \epsilon)$

- Lagrange multiplier $f(\mathbf{z}; t, \sigma) \equiv$ Vlasov distribution function.

◦ Note: all particle species have a kinetic description at this point.

Perturbed Action Functional $\mathcal{A}_\epsilon \equiv \sum_{n=1}^{\infty} \epsilon^n \mathcal{A}_n$

- nth-order action functional $\mathcal{A}_n(S_n, \Phi_n, \mathbf{A}_n; S_{n-1}, \Phi_{n-1}, \mathbf{A}_{n-1}; ...)$

$$\mathcal{A}_n \equiv \sum_{k=0}^{n-1} \mathcal{A}_n^k[S_{n-k}, \Phi_{n-k}, \mathbf{A}_{n-k}]$$

○ Time-dependent reference Vlasov-Maxwell $(f_0, \mathbf{E}_0, \mathbf{B}_0) \rightarrow$

$$\mathcal{A}_n^0 = \int d^6z\, dt\; f_0 \left(\frac{d_0 S_n}{dt} - \overline{h}_n \right) + \int \frac{d^3x}{4\pi}\, dt \left(\mathbf{E}_0 \cdot \mathbf{E}_n \;-\; \mathbf{B}_0 \cdot \mathbf{B}_n \right) \equiv 0$$

where $\overline{h}_n = q\,\Phi_n - q\mathbf{A}_n \cdot (\mathbf{p} - q\mathbf{A}_0/c)/mc \equiv q\,(\Phi_n - \mathbf{A}_n \cdot \mathbf{v}/c)$.

○ Perturbed Vlasov-Maxwell dynamics at order ϵ^{n-1}:

$$\begin{aligned}
&\mathcal{A}_1 \equiv 0 \\
&\mathcal{A}_2[S_1, \Phi_1, \mathbf{A}_1;\; f_0, h_0] \\
&\mathcal{A}_3[S_2, \Phi_2, \mathbf{A}_2;\; S_1, \Phi_1, \mathbf{A}_1;\; f_0, h_0] \\
&\vdots
\end{aligned}$$

Second-order (Linearized) Vlasov-Maxwell Theories

- Quadratic action functional $\mathcal{A}_2[S_1, \Phi_1, \mathbf{A}_1]$

$$\mathcal{A}_2 \;=\; \int \frac{d^3r\,dt}{8\pi}\left(|\mathbf{E}_1|^2 \;-\; |\mathbf{B}_1|^2\right) \;-\; \int d^6z\,dt\,\left(\frac{q^2\,f_0}{2\,mc^2}\right)\,|\mathbf{A}_1|^2 \\ +\; \int d^6z\,dt\,\left[\left\{S_1,\; f_0\right\}\left(\frac{1}{2}\,\frac{d_0S_1}{dt} \;-\; h_1\right)\right]$$

- Variational principle $\delta\mathcal{A}_2 \;\equiv\; 0$ with respect to $(\delta S_1, \delta\Phi_1, \delta\mathbf{A}_1)$

$$\left\{\left(\frac{d_0S_1}{dt} - h_1\right),\; f_0\right\} \;=\; 0 \;\rightarrow\; \frac{d_0S_1}{dt} \;=\; h_1$$

$$\frac{1}{4\pi}\,\nabla\cdot\mathbf{E}_1 \;-\; q\int\left\{S_1,\; f_0\right\}d^3p \;=\; 0$$

$$\frac{1}{4\pi}\left(\frac{1}{c}\frac{\partial\mathbf{E}_1}{\partial t} - \nabla\times\mathbf{B}_1\right) + \frac{q}{c}\int\left[\mathbf{v}\left\{S_1,\; f_0\right\} - \frac{qf_0\,\mathbf{A}_1}{mc}\right]d^3p = 0$$

Second-order Noether Equation

- Variational principle $\rightarrow$ Noether equation

$$\delta\mathcal{A}_2 \equiv \int \delta\mathcal{L}_2 \;\; d^3x\, dt = 0 \;\; \rightarrow \;\; \delta\mathcal{L}_2 \equiv \frac{\partial \mathcal{J}_2}{\partial t} \; + \; \nabla\cdot\boldsymbol{\Gamma}_2$$

  - Second-order action density

$$\begin{aligned} \mathcal{J}_2 &\equiv \frac{\partial\mathcal{L}_2}{\partial(\partial_t S_1)}\,\delta S_1 \; + \; \frac{\partial\mathcal{L}_2}{\partial(\partial_t \mathbf{A}_1)}\cdot\delta\mathbf{A}_1 \\ &= \frac{1}{2}\int \delta S_1 \; \left\{ S_1,\; f_0 \right\} d^3p \; - \; \frac{\delta\mathbf{A}_1\cdot\mathbf{E}_1}{4\pi\; c} \end{aligned}$$

  - Second-order action-density flux

$$\begin{aligned} \boldsymbol{\Gamma}_2 &\equiv \frac{\partial\mathcal{L}_2}{\partial(\nabla S_1)}\,\delta S_1 \; + \; \frac{\partial\mathcal{L}_2}{\partial(\nabla \Phi_1)}\,\delta\Phi_1 \; + \; \frac{\partial\mathcal{L}_2}{\partial(\nabla \mathbf{A}_1)}\cdot\delta\mathbf{A}_1 \\ &= \frac{1}{2}\int \delta S_1 \left(\frac{\partial h_0}{\partial\mathbf{p}} \; \left\{ S_1,\; f_0 \right\} - \frac{\partial f_0}{\partial \mathbf{p}}\; h_1 \right) d^3p \\ &\quad - \frac{1}{4\pi}\left(\delta\Phi_1\,\mathbf{E}_1 \; + \; \delta\mathbf{A}_1\times\mathbf{B}_1 \right) \end{aligned}$$

Noether Theorem: Energy Conservation Law (?)

- Invariance under time translations: $t \to t + \delta t$

$$
\begin{aligned}
\left(\delta S_1,\ \delta\Phi_1\right) &= -\,\delta t\left(\frac{\partial S_1}{\partial t},\ \frac{\partial\Phi_1}{\partial t}\right) \\
\delta\mathbf{A}_1 &= -\,\delta t\,\frac{\partial\mathbf{A}_1}{\partial t} \;=\; c\,\delta t\left(\mathbf{E}_1 \;+\; \nabla\Phi_1\right) \\
\delta\mathcal{L}_2 &= -\,\delta t\left(\frac{\partial\mathcal{L}_2}{\partial t} \;-\; \frac{\partial'\mathcal{L}_2}{\partial t}\right)
\end{aligned}
$$

- Energy transfer (perturbed particles-fields $\leftrightarrow$ reference)

$$
\frac{\partial\mathcal{E}_2}{\partial t} \;+\; \nabla\cdot\mathbf{S}_2 \;=\; -\,\frac{\partial'\mathcal{L}_2}{\partial t} \neq 0 \;\; (\text{if } f_0, \Phi_0, \mathbf{A}_0 \text{ are time-dependent})
$$

◦ Second-order free energy: $\{S_1, f_0\} = \{S_1, h_0\}\,\partial f_0/\partial h_0 + \cdots$

$$
\begin{aligned}
\mathcal{E}_2 &= \frac{1}{8\pi}\left(|\mathbf{E}_1|^2 \;+\; |\mathbf{B}_1|^2\right) \;-\; \frac{1}{2}\int\left\{S_1,\ f_0\right\}\,\left\{S_1,\ h_0\right\}d^3p \\
&\quad + \frac{q}{c}\,\mathbf{A}_1\cdot\int\left(\frac{q\,f_0}{2mc}\,\mathbf{A}_1 \;-\; \mathbf{v}\,\left\{S_1,\ f_0\right\}\right)\,d^3p
\end{aligned}
$$

Noether Theorem: Wave Action Conservation Law

- Field complexification $(S_1, \Phi_1, \mathbf{A}_1) \rightarrow (S_1, S_1^*, \Phi_1, \Phi_1^*, \mathbf{A}_1, \mathbf{A}_1^*)$

  ◦ Real-valued Lagrangian density $\mathcal{L}_{2R}$

$$\mathcal{L}_{2R} = \frac{1}{8\pi}\left(|\mathbf{E}_1|^2 - |\mathbf{B}_1|^2\right) - \left(\int \frac{q^2 f_0}{2\,mc^2}\,d^3p\right)|\mathbf{A}_1|^2 + \int \mathrm{Re}\left[\left\{S_1^*,\ f_0\right\}\left(\frac{1}{2}\,\frac{d_0 S_1}{dt} - h_1\right)\right] d^3p$$

  ◦ Phase variation $(\delta S_1, \delta S_1^*, ...) = i\,\delta\theta\,(S_1, -\,S_1^*, ...)$

- Wave-action conservation law $\delta\mathcal{L}_{2R} \equiv 0 = \partial_t \mathcal{J}_2 + \nabla\cdot\mathbf{\Gamma}_2$

  ◦ Wave-action density (Note: Case – van Kampen adjoint)

$$\mathcal{J}_2 = \int \mathrm{Im}\left(S_1^*\,\left\{S_1,\ f_0\right\}\right) d^3p - \mathrm{Im}\left(\frac{\mathbf{A}_1^*\cdot\mathbf{E}_1}{4\pi\,c}\right)$$

Particle (RE) Kinetic-MHD Equations (with Tronci)

- Particle Kinetic-MHD Lagrangian Density $\psi^a \equiv (S_1,\ \boldsymbol{\xi})$

$$\mathcal{L}_2 \;=\; \int \left\{ S_1,\ f_0 \right\} \left(\frac{1}{2}\,\frac{d_0 S_1}{dt} - H_1(\boldsymbol{\xi}) \right) + \frac{\rho_0}{2} \left| \frac{d_u \boldsymbol{\xi}}{dt} \right|^2 + \frac{1}{2} \boldsymbol{\xi} \cdot \mathbf{G}_u(\boldsymbol{\xi})$$

◦ Self-adjoint MHD operator $\mathbf{G}_u(\boldsymbol{\xi}) = \mathbf{F}_u(\boldsymbol{\xi}) + \nabla \cdot (\rho_0 \mathbf{u}_0 \mathbf{u}_0 \cdot \nabla \boldsymbol{\xi})$ $(d_u/dt = \partial/\partial t + \mathbf{u}_0 \cdot \nabla$ and $\mathbf{F}_u$ includes $|\mathbf{A}_1|^2$-contribution)

$$\int_{\mathbf{r}} \boldsymbol{\xi} \cdot \mathbf{G}_u(\delta\boldsymbol{\xi}) = \int_{\mathbf{r}} \delta\boldsymbol{\xi} \cdot \mathbf{G}_u(\boldsymbol{\xi})$$

◦ Perturbed particle Hamiltonian (MHD: $\mathbf{E}_0 \equiv -\,\mathbf{u}_0 \times \mathbf{B}_0/c$)

$$\left. \begin{array}{rcl} \Phi_1 & = & \boldsymbol{\xi} \cdot \mathbf{E}_0 \\ \mathbf{A}_1 & = & \boldsymbol{\xi} \times \mathbf{B}_0 \end{array} \right\} \rightarrow\ H_1(\boldsymbol{\xi}) = -\,\frac{q}{c}\,\boldsymbol{\xi} \cdot \Big((\mathbf{u}_0 - \ \mathbf{v}) \times \mathbf{B}_0 \Big)$$

◦ kinetic-MHD equation with RE kinetic current coupling

$$\frac{\partial}{\partial t} \left(\rho_0\, \frac{\partial \boldsymbol{\xi}}{\partial t} \right) \;=\; \mathbf{G}_u(\boldsymbol{\xi}) \;+\; \left[\frac{q}{c} \int (\mathbf{u}_0 - \mathbf{v})\ \left\{ S_1,\ f_0 \right\} d^3p \right] \times \mathbf{B}_0$$

Pressure coupling: Drift-kinetic/Gyrokinetic-MHD Models

- Particle → Guiding-center Quadratic Action Functional

$$
\begin{aligned}
\mathcal{A}_{2\mathrm{gc}} \;=\; & \int \frac{d^3r\,dt}{8\pi}\left(|\mathbf{E}_1|^2 \;-\; |\mathbf{B}_1|^2\right) \;-\; \int d^6Z\,dt\,\left(\frac{q^2\,F_0}{2\,mc^2}\right)\,|\mathbf{A}_{1\mathrm{gc}}|^2 \\
& +\; \int d^6Z\,dt\,\left[\left\{S_{1\mathrm{gc}},\;F_0\right\}_{\mathrm{gc}}\left(\frac{1}{2}\,\frac{d_{\mathrm{gc}}S_{1\mathrm{gc}}}{dt} \;-\; H_{1\mathrm{gc}}\right)\right]
\end{aligned}
$$

- First-order guiding-center Hamiltonian

$$
H_{1\mathrm{gc}} \;=\; q\,\Phi_{1\mathrm{gc}} \;-\; q\,\mathbf{A}_{1\mathrm{gc}}\cdot\mathsf{T}_{\mathrm{gc}}^{-1}(\mathbf{v}/c) \;\equiv\; q\,\psi_{1\mathrm{gc}}
$$

- First-order generating function (low-frequency decomposition)

$$
S_{1\mathrm{gc}} \;\equiv\; \langle S_{1\mathrm{gc}}\rangle \;+\; \widetilde{S}_{1\mathrm{gc}} \;\;\rightarrow\;\; \left\{\begin{array}{l}
\langle S_{1\mathrm{gc}}\rangle \;\equiv\; S_{1\mathrm{gy}} \\[1em]
\widetilde{S}_{1\mathrm{gc}} \;\equiv\; q\,(d_{\mathrm{gc}}/dt)^{-1}\;\widetilde{\psi}_{1\mathrm{gc}}
\end{array}\right.
$$

Gyrocenter Quadratic Action Functional

- Gyrocenter quadratic action functional $\mathcal{A}_{2\rm gy}[S_{1\rm gy}, \Phi_1, \mathbf{A}_1]$

$$\mathcal{A}_{2\rm gy} = \int d^6\overline{Z}\,dt \left[\{S_{1\rm gy},\, \overline{F}_0\}_{\rm gc} \left(\frac{1}{2}\,\frac{d_{\rm gc}}{dt}\,S_{1\rm gy} - \langle H_{1\rm gc}\rangle \right) \right] - \int d^6\overline{Z}\,dt\,\overline{F}_0\,H_{2\rm gy} + \int \frac{d^3r\,dt}{8\pi} \left(|\nabla_\perp \Phi_1|^2 - |\mathbf{B}_1|^2 \right)$$

  - Unperturbed gyrocenter Vlasov distribution $\overline{F}_0(\overline{\mathcal{E}}, \overline{\mu}, \overline{\mathbf{X}})$
  - Second-order gyrocenter (ponderomotive) Hamiltonian

$$H_{2\rm gy} = \frac{q^2}{2\,mc^2} \left\langle |\mathbf{A}_{1\rm gc}|^2 \right\rangle - \frac{q}{2} \left\langle \left\{ \widetilde{S}_{1\rm gc},\, \widetilde{\psi}_{1\rm gc} \right\}_{\rm gc} \right\rangle$$

- First-order gyrocenter Vlasov distribution

$$\overline{F}_1 \equiv \left\{ S_{1\rm gy},\, \overline{F}_0 \right\}_{\rm gc} = \left\{ S_{1\rm gy},\, \overline{\mathcal{E}} \right\}_{\rm gc} \frac{\partial \overline{F}_0}{\partial \overline{\mathcal{E}}} + \frac{c\widehat{\mathsf{b}}}{qB_\parallel^*} \times \overline{\nabla} \overline{F}_0 \cdot \overline{\nabla} S_{1\rm gy}$$

Nonadiabatic Gyrocenter Quadratic Action Functional

- Nonadiabatic part of $\overline{F}_1 \equiv \{S_{1\rm gy},\, \overline{F}_0\}_{\rm gc}$:

$$\begin{aligned} \overline{G}_1 \;&\equiv\; \overline{F}_1 \;-\; \frac{d_{\rm gc}\overline{S}_{1\rm gy}}{dt}\,\frac{\partial \overline{F}_0}{\partial \overline{\mathcal{E}}} \;\equiv\; \widehat{\mathcal{Q}}S_{1\rm gy} \\ &=\; \left(\frac{c\widehat{\mathsf{b}}}{qB_\|^*} \times \overline{\nabla}\overline{F}_0 \cdot \overline{\nabla} \;-\; \frac{\partial \overline{F}_0}{\partial \overline{\mathcal{E}}}\,\frac{\partial}{\partial t} \right) S_{1\rm gy} \end{aligned}$$

- Nonadiabatic gyrocenter quadratic action functional

$$\begin{aligned} \mathcal{A}_{2\rm gy} \;&=\; \int d^6\overline{Z}\,dt \left[\widehat{\mathcal{Q}}S_{1\rm gy} \left(\frac{1}{2}\,\frac{d_{\rm gc}}{dt}\,S_{1\rm gy} \;-\; \langle H_{1\rm gc}\rangle \right)\right] \\ &\quad -\; \int d^6\overline{Z}\,dt\;\overline{F}_0 \left(H_{2\rm gy} \;-\; \frac{1}{2}\,\frac{\partial \langle H_{1\rm gc}\rangle^2}{\partial \overline{\mathcal{E}}} \right) \\ &\quad +\; \int \frac{d^3r\,dt}{8\pi} \left(|\nabla_\perp \Phi_1|^2 - |\mathbf{B}_1|^2 \right) \end{aligned}$$

Linearized Drift-Kinetic-MHD Equations (Chen-White-Rosenbluth hybrid kinetic-MHD)

- Drift-Kinetic-MHD Lagrangian Density $\psi^a \equiv (S_{\rm dk},\ \boldsymbol{\xi})$

$$\mathcal{L} \;=\; \int d^2P \left[\frac{1}{2}\,\frac{d_{\rm gc}S_{\rm dk}}{dt} - H_{1\rm dk}(\boldsymbol{\xi}_\perp)\right]\widehat{\mathcal{Q}}S_{\rm dk} + \frac{\rho_0}{2}\left|\frac{\partial\boldsymbol{\xi}}{\partial t}\right|^2 + \frac{1}{2}\boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})$$

- ○ Self-adjoint MHD operator $\mathbf{F}(\boldsymbol{\xi})$ and Nonadiabatic operator

$$\widehat{\mathcal{Q}} \;\equiv\; \frac{\widehat{\mathsf{b}}}{qB}\times\nabla F_0\cdot\nabla \;-\; \frac{\partial F_0}{\partial\mathcal{E}}\,\frac{\partial}{\partial t}$$

- ○ Drift-kinetic Hamiltonian $(\Phi_1 = 0,\ \mathbf{A}_1 \equiv \boldsymbol{\xi}\times\mathbf{B})$

$$\begin{aligned}
H_{1\rm dk} &= -\,\frac{q}{c}\,\mathbf{A}_{1\perp}\cdot\mathbf{v}_{\rm gc} \;+\; \mu\,B_{1\|} \\
&= \mu\,\widehat{\mathsf{b}}\cdot\nabla\times(\boldsymbol{\xi}_\perp\times\mathbf{B}) \;+\; \boldsymbol{\xi}_\perp\cdot\left(\mu\,\nabla_\perp B \;+\; m\,v_\|^2\,\widehat{\mathsf{b}}\cdot\nabla\widehat{\mathsf{b}}\right) \\
&= \left(\mu\,B \;-\; mv_\|^2\right)\,\widehat{\mathsf{b}}\widehat{\mathsf{b}} : \nabla\boldsymbol{\xi}_\perp \;-\; \mu\,B\,\nabla\cdot\boldsymbol{\xi}_\perp
\end{aligned}$$

Pressure-coupled Drift-Kinetic-MHD Equations

- Euler-Lagrange Equations (Operators $d_{\rm gc}/dt$ and $\widehat{\mathcal{Q}}$ commute)

$$\delta S_{\rm dk} \;\;\rightarrow\;\; \frac{\partial S_{\rm dk}}{\partial t} \;+\; \Big\{ S_{\rm dk},\; H_0 \Big\}_{\rm gc} \;=\; H_{1{\rm dk}}(\boldsymbol{\xi}_{\perp})$$

$$\delta\boldsymbol{\xi} \;\;\rightarrow\;\; \rho_0\,\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} \;-\; \mathbf{F}(\boldsymbol{\xi}) \;=\; -\,\nabla\cdot\boldsymbol{\Pi}_1$$

- ◦ Nonadiabatic contribution to (CGL-like) pressure tensor due to energetic particles

$$\boldsymbol{\Pi}_1 \;=\; \int d^2P \left[\, \mu B \,(\mathbf{I} - \widehat{\sf b}\widehat{\sf b}) \;+\; m v_{\|}^2\, \widehat{\sf b}\widehat{\sf b} \,\right] \widehat{\mathcal{Q}} S_{\rm dk}$$

- ◦ Noether Method → Exact Conservation Laws (wave action)

Summary and Research Outlook

- Investigate particle and reduced kinetic-MHD model for RE

- Extend reduced kinetic-MHD model with $E_{1\|} \neq 0$

- Extend to nonlinear (cubic) perturbed action functional
  $\rightarrow$ Resonant three-wave interactions (Manley-Rowe relations)

$$
\begin{aligned}
\mathcal{L}_3 \;\; = \;\; & \frac{1}{3}\int \left[f_2 \left(\frac{d_0 S_1}{dt} - h_1 \right) + 2\, f_1 \left(\frac{d_0 S_2}{dt} - h_2 + \frac{1}{2}\left\{ S_1,\; h_1 \right\} \right) \right. \\
& \left. + \, f_0 \left(\left\{ S_1,\; h_2 \right\} + 2\left\{ S_2,\; h_1 \right\} - \frac{3q^2}{mc}\, \mathbf{A}_1 \cdot \mathbf{A}_2 \right) \right] \, d^3p \\
& + \frac{1}{4\pi} \left(\mathbf{E}_1 \cdot \mathbf{E}_2 \; - \; \mathbf{B}_1 \cdot \mathbf{B}_2 \right)
\end{aligned}
$$

◦ Second-order Vlasov distribution (with ponderomotive part)

$$
f_2 \; = \; \{S_2,\; f_0\} \; + \; \frac{1}{2}\left\{ S_1,\; \{S_1,\; f_0\} \right\}
$$