GUIDING-CENTER RADIATION-REACTION FORCE FOR RELATIVISTIC RUNAWAY ELECTRONS IN NONUNIFORM MAGNETIZED PLASMAS

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3rd Runaway Electron Meeting
Pertuis (France), June 17-19, 2015

OUTLINE

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IV. Summary

I. RADIATION-REACTION FORCE ON RELATIVISTIC RUNAWAY ELECTRONS

- **Kinetic equation for relativistic electrons** $(q = -e)$

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \{f,\, K\} + q\,\mathbf{E}\cdot\{\mathbf{x},\, f\} = \mathcal{R}[f] + \mathcal{C}[f]$$

- Relativistic Poisson bracket

$$\{F,\, G\} = \nabla F\cdot\frac{\partial G}{\partial \mathbf{p}} - \frac{\partial F}{\partial \mathbf{p}}\cdot\nabla G + \frac{q}{c}\mathbf{B}\cdot\frac{\partial F}{\partial \mathbf{p}}\times\frac{\partial G}{\partial \mathbf{p}}$$

- Relativistic kinetic energy

$$K = (\gamma - 1)\,mc^2 \quad \text{with} \quad \gamma = \sqrt{1 + |\mathbf{p}|^2/(mc)^2}$$

- Radiation-reaction (RR) force operator (particle-conserving) [Hirvijoki, Decker, Brizard, & Embreus (2015)]

$$\mathcal{R}[f] \;\equiv\; -\,\frac{\partial}{\partial \mathbf{p}}\cdot\left(\mathbf{F}_R\, f\right) \;=\; -\,\left\{x^i,\, F_R^i\, f\right\}$$

$$\boxed{\mathbf{F}_R \;=\; -\,\nu_R\,\left[\mathbf{p}_\perp \;+\; \left(\frac{|\mathbf{p}_\perp|^2}{(mc)^2}\right)\,\mathbf{p}\right]}$$

$$\epsilon_R \equiv \nu_R/\Omega_e \;=\; (2/3)\, r_e\, \gamma\Omega_e/c \;\simeq\; 10^{-12} \quad (B = 5\,\mathrm{T})$$

- Fokker-Planck collisional operator

$$\mathcal{C}[f] \;\equiv\; -\,\frac{\partial}{\partial \mathbf{p}}\cdot\left(\mathbf{F}_C\, f - \mathsf{D}_C\cdot\frac{\partial f}{\partial \mathbf{p}}\right) \;=\; -\,\left\{x^i,\, \left(F_C^i\, f - D_C^{ij}\{x^j,\, f\}\right)\right\}$$

- Ordering of RR-force versus Collisions

$$\frac{|\mathbf{F}_R|}{|\mathbf{F}_C|} \;\simeq\; \frac{2}{3}\,\frac{m_i}{m_e}\,\left(\frac{v_A}{c}\right)^2\,\frac{(\gamma^2-1)^{\frac{3}{2}}}{\gamma\;\ln\Lambda} \;\simeq\; \left(\frac{\gamma}{10}\right)^2$$

- **Magnetic Field Nonuniformity** ($\epsilon_B \equiv \rho/L_B = \epsilon_R\,\Delta$)

$$\Delta \;=\; \frac{\lambda_{\rm mfp}}{L_B} \;\rightarrow\; \left\{ \begin{array}{ll} \ll 1 & \text{(uniform magnetic field)} \\ \\ \gg 1 & \text{(nonuniform magnetic field)} \end{array} \right.$$

◦ Relativistic electrons ($\gamma = 20,\;\; B = 5\,$T)

$$\rho \;\simeq\; \frac{c}{\Omega}\,(\gamma^2 - 1)^{\frac{1}{2}} \;\simeq\; 17\,\frac{\gamma}{B({\rm G})}\;{\rm m} \;\;\rightarrow\;\; \sim\; 1\;{\rm cm}$$

$$\boxed{\rho/L_B \;\simeq\; 10^{-3} \;\gg\; \epsilon_R \;\;\rightarrow\;\; \Delta \gg 1}$$

- **Guiding-center transformation is used to obtain reduced kinetic equation for relativistic electrons**

◦ Guiding-center Fokker-Planck: Brizard (2004), Brizard *et al.* (2009), Decker *et al.* (2010), Hirvijoki *et al.* (2013)

◦ Guiding-center RR Force: Hirvijoki (2015)

II. DYNAMICAL REDUCTION BY GUIDING-CENTER PHASE-SPACE TRANSFORMATION

- **Lie-Transform Perturbation Theory**

∘ Near-Identity Transformation ($\epsilon \equiv \epsilon_B = \rho/L_B \ll 1$)

Particle $\mathbf{z} \leftrightarrow$ Guiding-center $\mathbf{Z} = \mathcal{T}_{\rm gc}\mathbf{z}$

$$
\begin{aligned}
Z^a(\mathbf{z};\epsilon) \;&=\; z^a \;+\; \epsilon\, G_1^a(\mathbf{z}) \\
&+\; \epsilon^2 \left[G_2^a(\mathbf{z}) \;+\; \frac{1}{2}\, G_1^b(\mathbf{z})\, \frac{\partial G_1^a(\mathbf{z})}{\partial z^b} \right] \;+\; \cdots
\end{aligned}
$$

where $(\mathbf{G}_1, \mathbf{G}_2, \cdots)$ are the generating vectors.

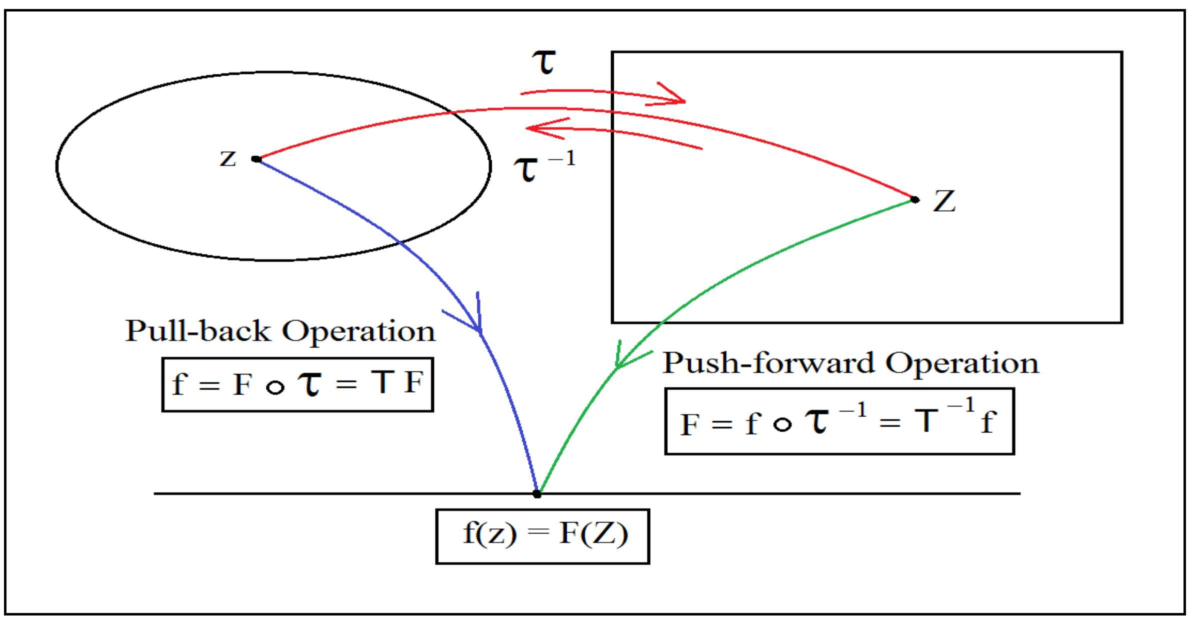
- **Guiding-center Operators**

Pull-Back Operator $\mathsf{T}_{\mathsf{gc}} : F \to f = \mathsf{T}_{\mathsf{gc}} F$

$$F(\mathbf{Z}) \;=\; F(\mathcal{T}_{gc}\,\mathbf{z}) \;=\; \mathsf{T}_{\mathsf{gc}} F(\mathbf{z}) \;=\; f(\mathbf{z})$$

Push-Forward Operator $\mathsf{T}_{\mathsf{gc}}^{-1} : f \to F = \mathsf{T}_{\mathsf{gc}}^{-1} f$

$$f(\mathbf{z}) \;=\; f(\mathcal{T}_{gc}^{-1}\mathbf{Z}) \;=\; \mathsf{T}_{\mathsf{gc}}^{-1} f(\mathbf{Z}) \;=\; F(\mathbf{Z})$$



- **Guiding-center Vlasov Operator**

$$\mathsf{T}_{\mathsf{gc}}^{-1}\left(\frac{df}{dt}\right) = \mathsf{T}_{\mathsf{gc}}^{-1}\left(\frac{d}{dt}\,\mathsf{T}_{\mathsf{gc}}F\right) = \frac{d_{\mathsf{gc}}F}{dt} \equiv \frac{\partial F}{\partial t} + \{F,\; K_{\mathsf{gc}}\}_{\mathsf{gc}} + q\,\mathsf{T}_{\mathsf{gc}}^{-1}\mathbf{E}\cdot\left\{\mathsf{T}_{\mathsf{gc}}^{-1}\mathbf{x},\; F\right\}_{\mathsf{gc}}$$

- Reduced Hamiltonian

$$K_{\mathsf{gc}} = m\,c^2\sqrt{1 + (2\mu\,B/mc^2) + (p_{\|}/mc)^2} - m\,c^2$$

- Reduced Poisson bracket $\{F,\; G\}_{\mathsf{gc}} \equiv \mathsf{T}_{\mathsf{gc}}^{-1}(\{\mathsf{T}_{\mathsf{gc}}F,\; \mathsf{T}_{\mathsf{gc}}G\})$

$$\{F,\; G\}_{\mathsf{gc}} \equiv \frac{\Omega}{\epsilon\,B}\left(\frac{\partial F}{\partial\theta}\frac{\partial G}{\partial\mu} - \frac{\partial F}{\partial\mu}\frac{\partial G}{\partial\theta}\right) + \frac{\mathbf{B}^*}{B_{\|}^*}\cdot\left(\nabla F\,\frac{\partial G}{\partial p_{\|}} - \frac{\partial F}{\partial p_{\|}}\nabla G\right) - \epsilon\,\frac{c\widehat{\mathsf{b}}}{qB_{\|}^*}\cdot\nabla F\times\nabla G$$

where $\mathbf{B}^* \equiv \nabla\times[\mathbf{A} + (c/q)\,p_{\|}\widehat{\mathsf{b}}]$ and $B_{\|}^* \equiv \widehat{\mathbf{b}}\cdot\mathbf{B}^*$.

- **Guiding-center Kinetic Equation**

$$\frac{d_{\rm gc}F}{dt} = \mathcal{R}_{\rm gc}[F] \equiv \mathsf{T}_{\rm gc}^{-1}\left(\mathcal{R}\left[\mathsf{T}_{\rm gc}\,F\right]\right)$$

  - Guiding-center distribution $F \equiv \mathsf{T}_{\rm gc}^{-1} f$
  - Guiding-center phase-space coordinates $\mathbf{Z} \equiv (\mathbf{X}, p_{\|}; \mu, \theta)$
  - Guiding-center Vlasov operator $d_{\rm gc}/dt \equiv d_{\rm R}/dt + \Omega\,\partial/\partial\theta$

- **Fast and Slow Orbital Time Scales** $(\Omega^{-1}\,d_{\rm R}/dt \ll \partial/\partial\theta)$

  - Guiding-center kinetic equation splits into two coupled kinetic equations: $F \equiv \langle F\rangle + \widetilde{F}$

∘ θ-averaged distribution $\langle F \rangle$

$$\frac{d_{\mathsf{R}}\langle F \rangle}{dt} = \langle \mathcal{R}_{\mathsf{gc}}[F] \rangle \equiv \langle \mathcal{R}_{\mathsf{gc}}[\langle F \rangle] \rangle + \left\langle \mathcal{R}_{\mathsf{gc}}[\widetilde{F}] \right\rangle$$

∘ θ-dependent distribution $\widetilde{F}$

$$\left(\Omega \frac{\partial}{\partial \theta} + \frac{d_{\mathsf{R}}}{dt} \right) \widetilde{F} = \mathcal{R}_{\mathsf{gc}}[F] - \langle \mathcal{R}_{\mathsf{gc}}[F] \rangle$$

∘ $\left\langle \mathcal{R}_{\mathsf{gc}}[\widetilde{F}] \right\rangle \neq 0 \Rightarrow \langle F \rangle$ and $\widetilde{F}$ are coupled by RR force.

- **Reduced Radiation-reaction Force Operator**

  - Guiding-center kinetic equation for $\langle F\rangle$ still exhibits θ-dependence through $\widetilde{F}$:

$$\frac{d_{\mathsf{R}}\langle F\rangle}{dt} \;-\; \langle\mathcal{R}_{\mathsf{gc}}[\langle F\rangle]\rangle \;=\; \left\langle\mathcal{R}_{\mathsf{gc}}[\widetilde{F}]\right\rangle \;\equiv\; \mathcal{O}\left(\frac{\nu_R}{\Omega}\right)$$

    Introduce dimensionless parameter $\epsilon_R = \nu_R/\Omega \ll 1$.

  - Solve for $\widetilde{F}[\langle F\rangle] = \epsilon_R\,\widetilde{F}_1[\langle F\rangle] + \cdots$

  - **Closure**: *Reduced* guiding-center kinetic equation

$$\boxed{\frac{d_{\mathsf{R}}\langle F\rangle}{dt} \;\equiv\; \mathcal{R}_{\mathsf{gc}}[\langle F\rangle]}$$

  - Zeroth-order reduced guiding-center RR-force operator

$$\boxed{\mathcal{R}_{\mathsf{gc}}[\langle F\rangle] \;\simeq\; \left\langle\mathsf{T}_{\mathsf{gc}}^{-1}\mathcal{R}[\mathsf{T}_{\mathsf{gc}}\langle F\rangle]\right\rangle}$$

IV. GUIDING-CENTER RADIATION-REACTION FORCE IN NONUNIFORM MAGNETIZED PLASMAS

- **Guiding-center radiation-reaction operator**

$$\mathcal{R}_{\rm gc}[F](\mathbf{Z}) \;=\; -\;\left\{X^{i}_{\rm gc}(\mathbf{Z}),\; F^{i}_{R\rm gc}(\mathbf{Z})\; F(\mathbf{Z})\;\right\}_{\rm gc}$$

- ◦ Guiding-center RR force

$$F^{i}_{R\rm gc}(\mathbf{Z}) \;=\; \mathsf{T}^{-1}_{\rm gc} F^{i}_{R}(\mathbf{Z})$$

- ◦ Guiding-center displacement $\boldsymbol{\rho}_{\rm gc} \equiv \mathsf{T}^{-1}_{\rm gc}\mathbf{x} - \mathbf{X}$

$$X^{i}_{\rm gc}(\mathbf{Z}) \;=\; X^{i} \;+\; \rho^{i}_{\rm gc}(\mathbf{Z}) \;\equiv\; \mathsf{T}^{-1}_{\rm gc} x^{i}$$

- **Reduced Guiding-center RR-force Operator**

$$\mathcal{R}_{\mathsf{gc}}[F](\mathbf{X}, p_{\|}, \mu) \;=\; -\,\frac{1}{\mathcal{J}_{\mathsf{gc}}}\frac{\partial}{\partial Z^{\alpha}}\left(\mathcal{J}_{\mathsf{gc}}\,\mathcal{F}^{\alpha}_{R\mathsf{gc}}\,F\right)$$

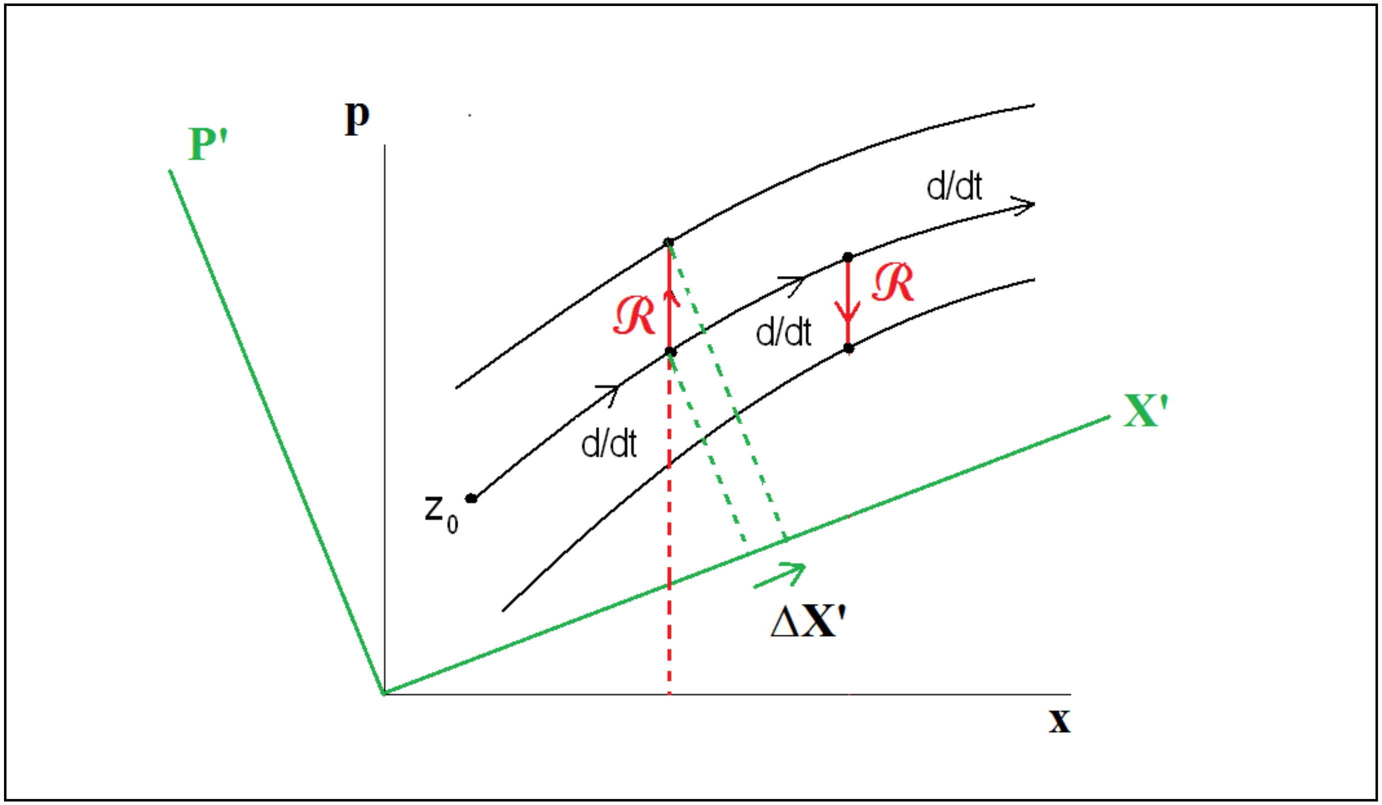
◦ Guiding-center phase-space RR-force components

$$\mathcal{F}^{\alpha}_{R\mathsf{gc}} \;=\; \left\langle \mathbf{F}_{R\mathsf{gc}} \cdot \mathbf{\Delta}^{\alpha}_{\mathsf{gc}} \right\rangle$$

◦ Guiding-center projection operators

$$\mathbf{\Delta}^{\alpha}_{\mathsf{gc}} \;=\; \{\mathbf{X}_{\mathsf{gc}},\; Z^{\alpha}\}_{\mathsf{gc}} \;\equiv\; \mathsf{T}^{-1}_{\mathsf{gc}}\left(\frac{\partial(\mathsf{T}_{\mathsf{gc}} Z^{\alpha})}{\partial \mathbf{p}}\right)$$

- **Local vs Guiding-center Radiation-reaction Force**



Spatial and Momentum Drag

- **Guiding-center radiation-reaction drag components**

◦ Guiding-center spatial drag velocity

$$\mathcal{F}_{R\mathrm{gc}}^{\mathbf{X}} = \left\langle \mathbf{F}_{R\mathrm{gc}} \right\rangle \times \frac{\widehat{\mathsf{b}}}{m\Omega_{\|}^{*}} + \left\langle \mathbf{F}_{R\mathrm{gc}} \cdot \left\{ \boldsymbol{\rho}_{\mathrm{gc}},\, \mathbf{X} \right\}_{\mathrm{gc}} \right\rangle$$

◦ Guiding-center parallel-momentum & magnetic-moment drag components

$$\mathcal{F}_{R\mathrm{gc}}^{p_{\|}} = \left\langle \mathbf{F}_{R\mathrm{gc}} \cdot \left[\frac{\mathbf{B}^{*}}{B_{\|}^{*}} \cdot \nabla \left(\mathbf{X} + \boldsymbol{\rho}_{\mathrm{gc}} \right) \right] \right\rangle$$

$$\mathcal{F}_{R\mathrm{gc}}^{\mu} = \left\langle \mathbf{F}_{R\mathrm{gc}} \cdot \frac{\Omega}{B} \frac{\partial \boldsymbol{\rho}_{\mathrm{gc}}}{\partial \theta} \right\rangle$$

- **Results valid up to First Order** (ϵ_B)

  [Hirvijoki, Decker, Brizard, and Embreus (2015)]

$$\nabla\times\widehat{\mathsf{b}} \equiv \tau_B\,\widehat{\mathsf{b}} + \widehat{\mathsf{b}}\times\boldsymbol{\kappa} \quad \text{and} \quad (\varrho_\|,\ \varrho_\perp) \equiv (p_\|/m\Omega, p_\perp/m\Omega)$$

- ○ Guiding-center spatial drag velocity

$$\mathcal{F}^{\mathbf{X}}_{R\mathsf{gc}} = -\,\epsilon_B\,\frac{\nu_R}{\Omega}\,\frac{2\,\mu B}{m\,c^2}\left(\widehat{\mathsf{b}}\times\dot{\mathbf{X}} + 3\,v_\|\,\varrho_\|\,\boldsymbol{\kappa}\right)$$

- ○ Guiding-center parallel-momentum drag

$$\mathcal{F}^{p_\|}_{R\mathsf{gc}} = -\,\nu_R\left[\frac{\mu B}{m\,c^2}\left(2 + \epsilon_B\,\varrho_\|\tau_B\right)\,p_\| + \epsilon_B\,\frac{\gamma^2}{2}\,p_\perp\varrho_\perp\tau_B\right]$$

- ○ Guiding-center magnetic-moment drag

$$\mathcal{F}^{\mu}_{R\mathsf{gc}} = -\,\nu_R\,\mu\left(1 + \frac{2\,\mu B}{m\,c^2}\right)\left(2 + \epsilon_B\,\varrho_\|\tau_B\right)$$

V. SUMMARY

- Lie-transform perturbation methods can be used to asymptotically eliminate fast time scales from collisionless, dissipative, & collisional kinetic equations.

- Simulation Applications:

  ◦ Bump formation in the runaway-electron tail [Decker, Hirvijoki, *et al.* (2015)] (see this meeting)

  ◦ Radiation-reaction induced non-monotomic features in runaway-electron distributions [Hirvijoki, Pusztai, *et al.* (2015)]