Theory of runaway electrons in ITER: Equations, important parameters, and implications for mitigation

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Citation: Physics of Plasmas (1994-present) **22**, 032504 (2015); doi: 10.1063/1.4913582
View online: http://dx.doi.org/10.1063/1.4913582
View Table of Contents: http://scitation.aip.org/content/aip/journal/pop/22/3?ver=pdfcov
Published by the AIP Publishing

PHYSICS OF PLASMAS 22, 032504 (2015)

Theory of runaway electrons in ITER: Equations, important parameters, and implications for mitigation

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(Received 17 December 2014; accepted 13 February 2015; published online 10 March 2015)

The plasma current in ITER cannot be allowed to transfer from thermal to relativistic electron carriers. The potential for damage is too great. Before the final design is chosen for the mitigation system to prevent such a transfer, it is important that the parameters that control the physics be understood. Equations that determine these parameters and their characteristic values are derived. The mitigation benefits of the injection of impurities with the highest possible atomic number Z and the slowing plasma cooling during halo current mitigation to $\gtrsim 40$ ms in ITER are discussed. The highest possible Z increases the poloidal flux consumption required for each e-fold in the number of relativistic electrons and reduces the number of high energy seed electrons from which exponentiation builds. Slow cooling of the plasma during halo current mitigation also reduces the electron seed. Existing experiments could test physics elements required for mitigation but cannot carry out an integrated demonstration. ITER itself cannot carry out an integrated demonstration without excessive danger of damage unless the probability of successful mitigation is extremely high. The probability of success depends on the reliability of the theory. Equations required for a reliable Monte Carlo simulation are derived. © *2015 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4913582]

I. INTRODUCTION

Any phenomenon that causes a dissipation of the plasma current in ITER on a time scale faster than tens of seconds can result in a large fraction of the plasma current being carried by relativistic runaway electrons. The effect and its importance to ITER were pointed out by Fleischmann and co-workers[1] in 1993. Rosenbluth and Putvinski[2] published more detailed calculations in 1997.

A successful ITER program implies a proof-of-principle demonstration that a tokamak can be routinely operated above a plasma current, $I_p \gtrsim 5$ MA, at which damage can become severe from the formation of a large runaway current of relativistic electrons. For example, the transfer of the current to relativistic electrons generally results in a significant change in the current profile, which can drive resistive tearing modes with the resistivity determined by the parameters of the cold background plasma, not the relativistic electrons,[3] Sec. VIII. The loss of magnetic surfaces through tearing modes could result in a sudden release of the relativistic electrons from the plasma to the chamber walls.

The required runaway avoidance and mitigation techniques cannot be developed empirically while operating at a high current. The danger to ITER is too great. Theory is required. This paper derives required equations, determines the critical parameters, and discusses the implications for the mitigation of runaway electrons on ITER.

A. Critical parameters

Four parameters determine whether a current transfer to relativistic electrons will occur and the magnitude of the current of relativistic electrons if the transfer does occur:

- The local loop voltage V_ℓ.

The local loop voltage[4] is the slippage of the poloidal magnetic flux outside of a surface that contains a given toroidal flux, $V_\ell \equiv (\partial\psi_p/\partial t)_{\psi_t}$, and is equal to the average value of $\int \eta \vec{j}\cdot d\vec{\ell}$ per toroidal circuit, where $d\vec{\ell}$ is the differential distance along a magnetic field line.

As discussed in Sec. II, *Minimum loop voltage for runaways*, the drag force on background electrons exceeds the acceleration of the parallel electric field in ITER for electrons of all energies when $V_\ell \lesssim 3n_b$ Volts, where n_b is the background electron density in units of $10^{20}/\mathrm{m}^3$. In many ITER plasma scenarios, $n_b \approx 1$. When the loop voltage is larger, it is potentially possible for the parallel electric field to accelerate to arbitrarily high energies electrons that have an initial kinetic energy $(\gamma-1)m_ec^2$ with $3n_b\gamma^2/(\gamma^2-1)$ Volts $> V_\ell$. These are called runaway electrons. The Lorentz factor is $\gamma \equiv 1/\sqrt{1-v^2/c^2}$ with v the electron speed and c the speed of light.

- The poloidal flux change ψ_{e-fold} required for an e-fold in the strength of the relativistic-electron current.

As discussed in Sec. III, *Avalanche effect*, the dominant effect of Coulomb scattering for most plasma phenomena is given by the cumulative effect of many small changes. Although the probability is smaller by the Coulomb logarithm $\ln\Lambda$, an energetic electron in a single collision can elevate a cold electron to a sufficiently high energy to run away. The effect is an exponential increase in the number of runaway electrons, which is called a runaway avalanche.

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When the loop voltage is significantly greater than the voltage required for runaway, an initial current carried by runaway electrons will e-fold each time the poloidal flux outside of a magnetic surface that contains a given toroidal flux changes by $\psi_{e-fold} \gtrsim \psi_c$. Where the required poloidal flux change for an e-fold under idealized assumptions, such as no scattering,[1,2] is $\psi_c \approx 2.3\,\mathrm{V}\cdot\mathrm{s}$ for ITER.

The magnitude of ψ_{e-fold} is inversely proportional to the minimum effective energy for runaway, which is sensitive to the pitch-angle distribution of newly energized electrons, which is itself dependent on the pitch-angle distribution of the runaway distribution as a whole. The pitch-angle distribution used by Rosenbluth and Putvinski[2] and by later authors is based on the assumption that the runaway electrons are moving in perfect alignment with the magnetic field. Section III derives the pitch and energy distribution of newly energized electrons for an arbitrary pitch and energy distribution of the runaways and explains why the Rosenbluth-Putvinski approximations may not have adequate accuracy.

- The ratio $e^{-\sigma_s}$ of the initial number of electrons above the critical energy for runaway, called seed electrons, to the number of relativistic electrons required to carry the full plasma current.

As discussed in Sec. IV, *Control of avalanche seed*, the obvious source of seed electrons is the tail of the pre-thermal-quench Maxwellian. This source, as well as seed electrons from electron current drive, can be reduced to insignificance if the cooling is sufficiently slow $\gtrsim$40 ms. Other sources for the required electron seed are found to be far from obvious.

- The initial poloidal magnetic flux ψ_0 between the magnetic axis and the chamber walls.

The poloidal flux consumption required to transfer the plasma current to relativistic electrons is $\sigma_s\psi_{e-fold}$. Assuming the chamber walls are perfectly conducting on the time scale of the current transfer, the remaining poloidal flux inside the plasma chamber is $\psi_r \equiv \psi_0 - \sigma_s\psi_{e-fold}$. The relativistic current is $I_r = \psi_r/L_r$, where L_r is the inductance of the plasma current in the profile that it assumes after the transfer. Appendix A, *Magnetic flux changes during runaway acceleration*, shows that $L_r \approx \mu_0 R$. Generally it is thought that up to about 3 MA of runaway current could be safely handled on ITER, which corresponds to $\psi_r \approx 20\,\mathrm{V}\cdot\mathrm{s}$.

B. Important time scales

Studies of the development of a runaway current must consider three time scales:

- The characteristic time τ_c required for a high energy electron to slow to a thermal energy.

As discussed in Sec. II, the time it takes a relativistic electron with kinetic energy $(\gamma-1)m_ec^2$ to slow to thermal energy of the background plasma is $\tau_s = (\gamma-1)\tau_c$, where $\tau_c \approx 22.7\,\mathrm{ms}/n_b$. Methods of reducing the number of seed electrons must operate on this time scale to be effective.

- The time τ_R required for the runaway distribution to reach a steady-state form.

Section VII, *Power balance for runaways* shows that after a relaxation time $\tau_R = 2(\gamma_r-1)\tau_c\ln\Lambda$ the distribution function for runaways assumes a steady-state form though exponentially increasing in magnitude. The effective kinetic energy required for runaway is $K_r = (\gamma_r-1)m_ec^2$. Avalanche theory is simplified when the runaway distribution assumes a steady state form early in the process. Approximations that were made by Rosenbluth and Putvinski[2] are not valid for times short compared to τ_R.

- The time τ_T required for the transfer of the plasma current to relativistic electrons.

The time to transfer the current, which is given by $\tau_T = (\sigma_s\psi_{e\text{-}fold})/V_\ell$, is in large part determined by the resistivity of the cold background plasma, which determines the loop voltage V_ℓ. That is, until a large fraction of the total plasma current is carried by runaway electrons, the development of the runaway current is a passive process, which is determined by the properties of the cold background plasma. The transfer time τ_T need not be long compared to the time τ_R required for the runaway distribution to reach a steady-state form.

C. Effects modifying critical parameters

Strategies to avoid the transfer of plasma current to relativistic electrons must be based on modifications to the critical parameters. Several effects either can or potentially could produce such modifications:

- Pitch-angle scattering

The addition of a species with a high atomic-number Z to the background plasma has important beneficial effects by both increasing K_r, the kinetic energy required for runaway, and ψ_{e-fold}, the poloidal flux change required per e-fold. This is discussed in Sec. V, *Kinetic equation and Coulomb scattering*.

- Direct losses of energetic electrons

When high energy electrons have a characteristic loss time τ_ℓ that is shorter than ψ_{e-fold}/V_ℓ, the avalanche mechanism becomes sub-dominant and the transfer of current to relativistic electrons cannot occur. There are two processes that could cause a rapid loss: (1) The break up of magnetic surfaces can allow high energy electrons to rapidly leave the plasma by moving along the magnetic field lines.[5] This effect is discussed below under mitigation strategies. (2) Pitch-angle scattering can place electrons into trapped particle trajectories, which can be poorly confined in a non-axisymmetric plasma, Sec. V.

- Synchrotron radiation

Highly relativistic electrons can lose power more rapidly through synchrotron radiation than from Coulomb drag on the background plasma. Section VI, *Synchrotron radiation*, derives the effective collision operator for electrons due to radiative losses. The derivation has subtleties that have not been adequately discussed in the literature. Generally, synchrotron radiation should not greatly modify the transfer of current to runaway electrons in ITER though it can be important as a diagnostic tool.[6,28]

- Microturbulence

Microturbulence could in principle enhance pitch-angle scattering and energy loss—both are beneficial. Nevertheless, as discussed in Sec. IX, *Microturbulence*, existing theories do not give cause for optimism that this will occur. Since the effects of microturbulence make mitigation easier if they arise, ignoring the effects of microturbulence in a theory of mitigation is a reasonable conservative assumption.

D. Mitigation strategies

Not only will any naturally arising process that causes a rapid cooling of the plasma tend to cause a transfer of the plasma current to relativistic electrons but also the halo-current mitigation system will drive this transfer. For halo-current mitigation, the plasma current should be ramped down significantly faster than the wall time, which in ITER with $\psi_0 \sim 100\mathrm{V}\cdot\mathrm{s}$ implies $\psi_0/V_\ell \lesssim 150\,\mathrm{ms}$ and a loop voltage $V_\ell \gtrsim 700$ V. This voltage is far above the critical voltage for runaway unless the background density is increased 100's of times, which may not be possible and produces problems of its own.

The operational lifetime of ITER could be set by the effectiveness of the runaway-mitigation system. Several strategies for improving the runaway-mitigation systems are possible and two could probably be implemented on ITER.

- The use of slow plasma cooling, $\sim$40 ms, for halo mitigation.

Slow cooling gives adequate time for the high-energy electrons to slow, which allows a Maxwellian distribution to be maintained as the plasma cools and eliminates the large number of high energy electrons present during lower hybrid current drive. Rapid cooling produces a non-Maxwellian tail that serves as a runaway seed.

The plasma cooling time should not be confused with the time scale for the plasma current decay. The rate of current decay is determined by the electron temperature—not the rate of cooling—through the plasma resistivity. It may be beneficial to preemptively cool plasmas that are in high danger of disruption to avoid the rapid plasma cooling of a natural thermal quench.

Existing experiments could test and demonstrate some of the benefits of slow cooling, but the time required for magnetic field penetration through the surrounding walls and plasma densities differ between existing experiments from those in ITER in a way that complicates a comprehensive test.

- The use of higher atomic number, Z, impurities for runaway mitigation, the higher the Z the better.

The enhanced pitch-angle scattering of the high-energy electrons: (1) increases the energy required for runaway, which reduces the number of seed electrons; (2) increases the change in poloidal flux required for an e-fold in the number of runaways; and (3) reduces the radiative loss for a given electron drag. Deeply trapped electrons in weakly ionized high-Z atoms contribute to runaway drag but have a negligible radiative power loss.

A potential disadvantage of high-Z impurities is that the average energy of the runaway electrons is increased, but this disadvantage is of little importance if the runaway current is reduced exponentially in magnitude.

- Ensure the magnetic surfaces are destroyed whenever the loop voltage is large.

Magnetic surfaces may be destroyed during rapid thermal quenches. But even a narrow annulus of magnetic surfaces can confine relativistic electrons in ITER due to their small gyroradius. It is neither obvious that all magnetic surface are destroyed nor that they remain destroyed. It may be possible to design tokamaks, so the induction current in the walls that occurs when the plasma current rapidly drops would ensure the required breakup of surfaces, Sec. X. This was not done for ITER, and the only theoretical studies[7,8] have used highly simplified models.

II. MINIMUM LOOP VOLTAGE FOR RUNAWAYS

The parallel electric field, $E_{||}$, must be sufficiently large for runaway electrons to persist. The lowest value for this $E_{||}$ is called the critical electric field, E_c, which is calculated assuming that the only non-ideal effect on high-energy electrons is the Coulomb drag of the cold background electrons.

E_c is not an electric field; eE_c is the minimum drag force exerted on an electron by the background electrons. The expression for E_c is given in Eq. (12) of Connor and Hastie[9]

$$E_c = \frac{4\pi n_b e^3 \ln\Lambda}{m_e c^2} = \frac{n_b e^3 \ln\Lambda}{4\pi\epsilon_0^2 m_e c^2} \approx 0.075 n_b \tag{1}$$

in Gaussian and in Standard International units, where $\ln\Lambda$ is the Coulomb logarithm. The number density of background electrons n_b has units of $10^{20}/\mathrm{m}^3$ when the critical electric field has units of Volts per meter. A typical density for projected ITER operations is $n_b \approx 1$. The drag force on an electron that has a velocity v is

$$f_{drag} = \frac{c^2}{v^2} eE_c = \frac{\gamma^2}{\gamma^2 - 1} eE_c. \tag{2}$$

Only relativistic electrons experience the minimum force.

In addition to drag, Coulomb collisions with the background plasma cause pitch angle scattering, which will be discussed in Sec. V. Section V will also give the relativistically correct collision operator for high-energy electrons colliding with a cold background plasma from which the c^2/v^2 factor in the drag force is derived.

No electrons can runaway unless the component of the electric field along the magnetic field lines satisfies $E_{||} > E_c$. As demonstrated in Appendix A, this is equivalent to the requirement that the local loop voltage V_ℓ satisfies

$$V_\ell \geq V_c \approx 2\pi R E_c \approx 3 n_b \frac{R}{R_{ITER}} \text{Volts}, \tag{3}$$

where R is an average major radius of a magnetic surface and R_{ITER} is the major radius of ITER, 6.2 m.

The local loop voltage[4] is the rate of change of the poloidal magnetic flux ψ_p outside of a surface that contains a given toroidal flux, ψ_t

$$V_\ell \equiv \left(\frac{\partial \psi_p}{\partial t}\right)_{\psi_t}. \tag{4}$$

When magnetic surfaces do not exist, the relevant loop voltage is an average over the volume covered by a single magnetic field line, Eq. (A12).

The quantity of poloidal flux enclosed by the magnetic axis and available to produce runaways depends on the initial current profile and whether the flux outside the conducting structures that surround the plasma can penetrate those structures on the time required to exacerbate the runaway. Nevertheless as discussed in Appendix A, the available flux is approximately $\mu_0 R I_p = 117\,\text{V}\cdot\text{s}$ when the plasma current is 15 MA in ITER. Consequently, runaways cannot be produced in ITER when the current changes on a time scale longer than $\mu_0 R I_p / V_c \sim 40\,\text{s}/n_b$. Stated differently, the possibility of runaway production must be considered when the current in ITER changes rapidly compared to a 40 s time scale.

The background electrons drag down the kinetic energy of relativistic electrons, which means $\gamma - 1 \gg 1$, on the time scale

$$\tau_s = (\gamma - 1)\tau_c, \tag{5}$$

where

$$\tau_c \equiv \frac{m_e c}{e E_c} \approx \frac{22.7\text{ms}}{n_b}. \tag{6}$$

When the loop voltage V_ℓ is larger than V_c, the drag on background electrons, Eq. (2), assures that electrons with kinetic energy less that $(\gamma_c - 1) m_e c^2$ with $(\gamma_c^2 - 1)/\gamma_c^2 = V_c/V_\ell$ cannot runaway. For $V_\ell \gg V_c$

$$\gamma_c - 1 = \frac{V_c}{2V_\ell}. \tag{7}$$

III. AVALANCHE EFFECT

The standard Coulomb collision operator for electrons gives the diffusive effect of a large number small changes in the momentum of an electron due scattering from the rest of the plasma. The cross section for electron-electron scattering, the Møller cross section[10,11] when quantum and relativistic effects are included, is singular for small changes in the electron momentum. The resolution of this singularity gives the Coulomb logarithm that appears in E_c.

Finite changes in the momentum of an electron due to Møllor scattering are subdominant by a factor of $1/\ln\Lambda$ in most collisional processes in plasmas but are a critical element in electron runaway in the presence of a large change in the poloidal magnetic flux $\Delta\psi_p$. The importance of a single scattering event converting a cold into a runaway electron to ITER was pointed out by Fleischmann and co-workers,[1] but the effect is usually called the Rosenbluth avalanche due to the more detailed treatment of Rosenbluth and Putvinski.[2] As will be discussed in this section, this kind of scattering can cause the number of runaways to increase as $\exp(|\Delta\psi_p|/\psi_{e-fold})$, where $\psi_{e-fold} \gtrsim \psi_c \approx 2.3\text{V}\cdot\text{s}$ in a tokamak the size of ITER. The exponential increase is called the runaway avalanche.

The source function for newly energized electrons $S(p,\lambda)$ is defined so, if that were the only term, the kinetic equation for energetic electrons would be $\partial f/\partial t = S$. The source function $S(p,\lambda)$ for a newly energized electron of momentum p and pitch λ is the result of a collision of an electron of negligible energy with an energetic electron that had momentum p_e and pitch λ_e before the collision. That is,

$$S(p,\lambda) = \frac{\int \mathcal{S}(p_e,\lambda_e,p,\lambda) f(p_e,\lambda_e) 2\pi p_e^2 dp_e d\lambda}{2\pi p^2}. \tag{8}$$

The function $\mathcal{S}(p_e,\lambda_e,p,\lambda)$ gives rate of change in the number of electrons within the differential distance $dp d\lambda$ of the momentum space position (p,λ) due to collisions with electrons that were within the differential distance $dp_e d\lambda_e$ of the momentum space position (p_e,λ_e) before the collision.

The momentum p, the pitch $\lambda \equiv \cos\vartheta$, and the gyrophase φ_p are spherical momentum coordinates. They are related to Cartesian momentum coordinates by $p_x = p\sin\vartheta\cos\varphi_p$, $p_y = p\sin\vartheta\sin\varphi_p$, and $p_z = p\cos\vartheta$. The phase angle φ_p, which advances at the rapid gyrofrequency, will be averaged over. After averaging, a momentum-space position is specified by (p,λ), and the momentum-space p volume element is

$$d^3p = 2\pi p^2 dp d\lambda. \tag{9}$$

By the definition of spherical coordinates, $\sin\vartheta = \sqrt{1-\lambda^2}$ is always positive.

The function $\mathcal{S}(p_e,\lambda_e,p,\lambda)$ is determined by $\mathcal{S}_0(p_e,\lambda_e,p,\lambda)$, which gives the rate at which an electron with momentum and pitch (p_e,λ_e) takes an electron with negligible kinetic energy to a momentum and pitch (p,λ) in a single collision. While doing this, energy and momentum conservation implies the phase-space position of the energetic electron is changed to (p_n,λ_n), where p_n and λ_n are functions of $(p_e,\lambda_e,p,\lambda)$, so

$$\begin{aligned}\mathcal{S} = (1 - \delta(p-p_e)\delta(\lambda-\lambda_e) \\ + \delta(p-p_n)\delta(\lambda-\lambda_n))\mathcal{S}_0(p_e,\lambda_e,p,\lambda).\end{aligned} \tag{10}$$

Energy conservation implies $\gamma_n(p_n) = 1 + \gamma_e(p_e) - \gamma(p)$ with the Lorentz factor written as $\gamma(p) = \sqrt{1+(p/m_e c)^2}$. Momentum conservation implies $\lambda_n p_n = \lambda_e p_e - \lambda p$. These are solvable equations for (p_n,λ_n) as functions of $(p_e,\lambda_e,p,\lambda)$. Momentum conservation in the two directions orthogonal to the magnetic field is irrelevant due to the average over the rapid gyromotion. $\mathcal{S}_0(p_e, \lambda_e, p, \lambda)$ is determined by two functions,

$$\mathcal{S}_0(p_e, \lambda_e, p, \lambda) = n_b v_e \frac{d\sigma_M(p_e, p)}{dp} \Pi(p_e, \lambda_e, p; \lambda), \tag{11}$$

where n_b is the number of cold background electrons.

The first function that appears in $\mathcal{S}_0(p_e, \lambda_e, p, \lambda)$ is the rate per energetic electron at which collisions occur that result in the energization of an electron to a momentum in the range dp about p. This rate is given by $v_e d\sigma_M/dp$, where $v_e = p_e/\gamma_e m_e$ is the speed of the energetic electron and $d\sigma_M$ is the differential Møllor cross section in the form given by Ashkin *et al.*[11]

$$d\sigma_M(\gamma_e, \nu) = 2\pi r_o^2 \frac{\gamma_e^2}{(\gamma_e - 1)^2(\gamma_e + 1)} \times \left(x^2 - 3x + \left(\frac{\gamma_e - 1}{\gamma_e}\right)^2 (1 + x) \right) d\nu, \tag{12}$$

$$x \equiv \frac{1}{\nu(1 - \nu)}, \tag{13}$$

$$\nu \equiv \frac{\gamma - 1}{\gamma_e - 1}. \tag{14}$$

The classical radius of an electron, r_o, is defined by $e^2/(4\pi\epsilon_0 r_o) = m_e c^2$. That is, $2\pi r_0^2 = eE_c/(2m_e c^2 n_b \ln\Lambda)$. The Lorentz factors are γ for the newly energized electron after the collision and γ_e for the energetic electron before the collision. The fractional transfer of kinetic energy is ν.

The second function that appears in $\mathcal{S}_0(p_e, \lambda_e, p; \lambda)$ is the probability $\Pi(p, \lambda, p_e; \lambda_e)$ that the newly energized electron will have a pitch in the range $d\lambda$ after the collision given the momentum p_e and pitch λ_e of the energetic electron before the collision and the momentum p of the energized electron after the collision. Since Π is a probability, $\int_{-1}^{1} \Pi d\lambda = 1$. Section III B shows that four-momentum conservation implies

$$\Pi(p_e, \lambda_e, p; \lambda) = \frac{1}{\pi\sqrt{\lambda_2^2 - (\lambda - \lambda_1)^2}}, \tag{15}$$

$$\lambda_1(p, p_e, \lambda_e) \equiv \sqrt{\frac{(\gamma_e + 1)(\gamma - 1)}{(\gamma_e - 1)(\gamma + 1)}} \lambda_e, \tag{16}$$

$$\lambda_2(p, p_e, \lambda_e) \equiv \sqrt{\frac{2(\gamma_e - \gamma)}{(\gamma_e - 1)(\gamma + 1)} \left(1 - \lambda_e^2\right)}, \tag{17}$$

when $|\lambda - \lambda_1| \leq \lambda_2$. Otherwise, $\Pi(p_e, \lambda_e, p; \lambda) = 0$.

When a significant fraction of the background electrons are bound in partially ionized ions, the binding energy and the momentum of the bound electrons must be taken into account unless either their binding energy is small compared to kinetic energies required for runaway or the binding energy is large compared to the energy of the energetic electrons. A Thomas-Fermi model of ions would give a reasonable approximation, but here it will be assumed for simplicity that the binding energies are small compared to the energy required for runaway.

A. Approximate $\mathcal{S}$ of Rosenbluth-Putvinski

Rosenbluth and Putvinski[2] gave an approximate expression for $\mathcal{S}$, which is the standard expression in the literature on the avalanche effect. They made three approximations: (1) the fractional transfer of kinetic energy is small, $\nu \ll 1$, (2) the energetic electron is relativistic, $\gamma_e - 1 \gg 1$, and (3) the momentum of energetic electron is well aligned with the magnetic field, $|\lambda_e| \to 1$.

When the fractional energy transfer is small, $\nu \to 0$, the change in the energy and momentum of the energetic electron is negligible, so $\mathcal{S} \to \mathcal{S}_0$. In addition, the x^2 term dominates $d\sigma_M$. That and the second approximation, $\gamma_e - 1 \gg 1$, imply

$$v_e d\sigma_M = \frac{eE_c c}{2n_b \ln\Lambda} \frac{dK_s}{K_s^2}, \tag{18}$$

where $K_s = (\gamma - 1)m_e c^2$ is the kinetic energy of the newly energized electron. In the relativistic limit $\gamma_e - 1 \gg 1$, both $v_e d\sigma_M/dp$ and Π are independent of the momentum of the energetic electron, p_e The third approximation, $|\lambda_e| \to 1$, which is equivalent to the limit $\lambda_2 \to 0$, implies $\Pi(p_e, \lambda_e, p; \lambda) = \delta(\lambda - \lambda_1)$, a delta function about λ_1, which becomes independent of γ_e in the limit $\gamma_e - 1 \gg 1$.

The three Rosenbluth-Putvinski approximations make the source function $S(p, \lambda)$ proportional to the number density of runaway electrons, independent of the details of the distribution function $f(p, \lambda)$, which simplifies calculations. The number of runaway electrons n_r increases as $\int S 2\pi p^2 dp d\lambda$, or

$$\frac{d\ln n_r}{dt} = \frac{ecE_c}{2\ln\Lambda} \frac{1}{K_r}, \tag{19}$$

where K_r is the effective kinetic energy required for runaway.

1. Effective energy for runaway

The primary subtlety in Eq. (19) for runaway exponentiation is the appropriate value for the effective kinetic energy for runaway $K_r = (\gamma_r - 1)m_e c^2$. The electric-field acceleration exceeds Coulomb drag for non-relativistic electrons with a velocity v perfectly aligned with the magnetic field when $(v/c)^2 > E_c/E_{||}$. Nevertheless, Jayakumar *et al.*[1] used $v^2/c^2 = 2E_c/E_{||}$ as the non-relativistic runaway condition as did Rosenbluth and Putvinski[2] in their Eq. (6). As recognized by Jayakumar *et al.*, the precise determination of the effect K_r for use in Eq. (19) has subtleties and is changed by pitch-angle scattering. Here a formula equivalent to Eq. (7) of Rosenbluth and Putvinski[2] will be used when pitch-angle scattering is ignored

$$\frac{d\ln n_r}{dt} = \frac{e\left(E_{||} - E_c\right)}{2m_e c \ln\Lambda} = \frac{V_\ell - V_c}{\psi_c}, \tag{20}$$

$$\psi_c \equiv 4\pi R \frac{m_e c}{e} \ln\Lambda \approx 2.32 \frac{R}{R_{ITER}} \mathrm{V \cdot s}. \tag{21}$$

Energy conservation, Sec. VII A, implies that, even though exponentially increasing in magnitude, the distribution function for the energetic electrons relaxes to a steady-state form with an average Lorentz factor

$$\bar{\gamma} - 1 \approx 2 \ln \Lambda \tag{22}$$

when $j_{||} \approx en_r c$, a result given by Rosenbluth and Putvinski.[2] This expression for $\bar{\gamma}$ is consistent with the first two of the Rosenbluth-Putvinski assumptions, $(\bar{\gamma} - 1)m_e c^2 \gg K_r$ and $\bar{\gamma} - 1 \gg 1$. Under many conditions the average pitch angle of the runaways is small, $|\delta\vartheta| \approx 1/(2 \ln \Lambda)$, Eq. (84), which is consistent with the third Rosenbluth-Putvinski assumption.

When $\bar{\gamma} - 1 \gg K_r$, negligible energy is required to create a new runaway, but significant energy is required to accelerate the newly created runaway electron from γ_r to $\bar{\gamma}$. The required poloidal flux change ψ_c to obtain an e-fold is given by flux change required to bring the energy of a newly energized electron to the average energy, $\bar{\gamma} - 1 \approx 2 \ln \Lambda$, of the runaway distribution, which can be calculated using the conservation of toroidal canonical momentum $P_\varphi = \gamma R m_e v_\varphi + e\psi_p / 2\pi$.

When the loop voltage is large compared to the critical voltage, $V_\ell \gg V_c$, Eq. (20) predicts the density of runaway electrons n_r increases from its initial, or seed, value n_s as

$$n_r = n_s \exp\left(\frac{|\Delta\psi_p|}{\psi_c}\right), \tag{23}$$

where $|\Delta\psi_p|$ is the change in the poloidal magnetic flux outside a surface that encloses a given amount of toroidal magnetic flux.

The transfer from an Ohmic current carried by thermal electrons to runaway current is on a time-scale set by the Ohmic loop voltage. When $|\Delta\psi_p|_{req}$ is the change in the poloidal flux required to obtain necessary number of e-folds, the required time for the transfer is

$$\tau_T = \frac{|\Delta\psi_p|_{req}}{V_\ell}. \tag{24}$$

The electric field drops to critical field for producing runaways when the full plasma current is carried by runaways.

Although $\bar{\gamma} - 1 \approx 2 \ln \Lambda$, the maximum kinetic energy $(\gamma_{max} - 1)m_e c^2$ that an electron can reach is much higher. Using results for Appendix A, one can show $\gamma_{max} = e|\Delta\psi_p| / (2\pi R m_e c) = \bar{\gamma}|\Delta\psi_p|/\psi_c$.

Any uncertainty in the effective energy for runaway K_r translates into an uncertainty in the average kinetic energy of runaway electrons $(\bar{\gamma} - 1)m_e c^2$, Eq. (82), and hence the poloidal flux change ψ_{e-fold} required for an e-fold in the number of runaway electrons. An uncertainty in ψ_{e-fold} implies an uncertainty in the poloidal flux $|\Delta\psi_p|_{req}$ that must be consumed to transfer the current from thermal to relativistic electrons, and therefore the magnitude of the current of relativistic electrons. The calculation of of K_r requires an accurate source function for newly energized electrons.

2. Limitations of the Rosenbluth-Putvinski approximation

Although the assumptions that give the Rosenbluth-Putvinski approximation to $\mathcal{S}$ appear to hold in a number of situations, the approximation may not give an accurate expression for the poloidal flux change required to transfer the plasma current into a relativistic runaway current.

As discussed in Sec. VII, a long time may be required for the runaway distribution to reach its steady, though exponentially increasing, form. During this period, the Rosenbluth-Putvinski assumptions are not generally valid, and a large error can result in the poloidal flux change required to transfer the current from near-thermal to runaway electrons. During the early phases of the avalanche, a large fractional change in the energy of the energetic electrons may be required to produce additional runaways by the avalanche mechanism and the typical runaway electron may not be strongly relativistic.

Under many conditions the average pitch angle of the runaways is small, $|\delta\vartheta| \approx 1/(2 \ln \Lambda)$, Eq. (84), as assumed by Rosenbluth and Putvinski. However, a newly energized electron, when non-relativistic, moves almost perpendicular to the momentum of the energetic electron that produced it, which means almost perpendicular to the magnetic field lines. Such electrons gain energy only slowly from the parallel electric field and are prone to magnetic trapping, which prevents gaining energy from the parallel electric field. The Rosenbluth-Putvinski approximation is that $\Pi(p_e, \lambda_e, p; \lambda) \approx \delta(\lambda - \lambda_1)$ with $\lambda_1 \approx \sqrt{(\gamma - 1)/(\gamma + 1)}$. When the newly energized electron is not relativistic, its pitch is $|\lambda| = v/2c$. The spread in pitch angle of subrelativistic newly energized electrons about $\lambda = \lambda_1$ is given by $\lambda_2 \approx \delta\vartheta_e$, where $\delta\vartheta_e$ is the spread in pitch of the energetic electrons. Especially in the early stages of runaway, $\delta\vartheta_e$ need not be small. The effective kinetic energy of runaway K_r is determined by the behavior of newly energized electrons, and the required flux change to transfer the current is inversely proportional to K_r.

B. Scattering probability $\Pi(p_e, \lambda_e, p; \lambda)$

Electron-electron scattering is energy and momentum conserving. When two electrons collide, one with a far larger kinetic energy than the other, energy and momentum conservation determine the probability $\Pi(p_e, \lambda_e, p; \lambda)$ that the electron that is energized will have a pitch in the range $d\lambda$ after the collision given the momentum p_e and pitch λ_e of the energetic electron before the collision and the momentum p of the energized electron after the collision.

1. Scattering angle

An important element in the derivation of Π is the angle θ_δ between the velocity vector of secondary electron and the velocity vector of the high energy electron that was the other participant in the collision.

The properties of the Lorentz transformation imply the angle θ_δ is uniquely determined by the kinetic energy, $K_s = (\gamma - 1)m_e c^2$, imparted to the secondary electron. This section will use these properties to show the secondary electrons move on the surface of a cone oriented around the velocity direction of the high energy electron with the opening angle of the cone θ_δ given by Eq. (31)

When the fractional energy transfer ν is small and the secondary electron is sub-relativistic, $\gamma - 1 \ll 1$, Eq. (31) implies the secondary electron moves in an essentially orthogonal direction to the direction of the high energy electron. When the secondary electron is itself strongly relativistic, it moves in essentially the same direction as the high energy electron.

In relativity theory, the four momentum of an electron is $p^\mu = (c, \vec{v})\gamma m_e$ and $p_\mu = (-c, \vec{v})\gamma m_e$. The index μ, which runs from zero to three, numbers the components. The time-like component of a four vector changes sign when the index μ is changed from a superscript to a subscript. The dot product of any two four vectors—one must be superscript and the other in subscript form—is a Lorentz invariant, so $\sum_\mu p^\mu p_\mu = -(c^2 - v^2)\gamma^2 m_e^2 = -m_e^2 c^2$, which implies $\gamma^2 = 1/(1 - v^2/c^2)$, the well known expression for the relativistic Lorentz factor.

When the frame of reference is changed so the new frame is moving with a velocity $-v_f\hat{x}$ relative to the original frame, the four momentum, or any other four vector, in new frame p_f^ν is related to the four vector in the old frame by $p_f^\nu = \sum_\mu \Lambda_\mu^\nu p^\mu$, where the Lorentz transformation matrix is

$$\Lambda_\mu^\nu = \begin{pmatrix} \gamma_f & -\gamma_f \frac{v_f}{c} & 0 & 0 \\ -\gamma_f \frac{v_f}{c} & \gamma_f & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (25)$$

and $\gamma_f = 1/\sqrt{1 - v_f^2/c^2}$.

When an energetic electron with four momentum $p^\mu = (c, v_e\hat{x})\gamma_e m_e$ strikes a background electron with four momentum $q^\mu = (c, \vec{0})m_e$, the center-of-momentum frame is defined so $p_f^1 + q_f^1 = 0$. That is, the $\hat{x}$ component of the sum of the two momenta vanishes. The Lorentz transformed momenta are $p_f^1 = (v_e - v_f)\gamma_f \gamma m_e$ and $q_f^1 = -v_f\gamma_f m_e$, so $(v_e - v_f)\gamma - v_f = 0$. The speed and the Lorentz factor of the center-of-momentum frame are determined by the Lorentz factor, γ_e, of the energetic electron

$$\left(\frac{v_f}{c}\right)^2 = \frac{\gamma_e - 1}{\gamma_e + 1}, \quad (26)$$

$$\gamma_f = \sqrt{\frac{\gamma_e + 1}{2}}. \quad (27)$$

A bared q^μ denotes the post-collision four momentum of the newly energized electron. In the center-of-momentum frame

$$\bar{q}_f^\mu = \left(1, -\frac{v_f}{c}\cos\theta_c \hat{x}, -\frac{v_f}{c}\sin\theta_c \hat{y}, 0\right)\gamma_f m_e c, \quad (28)$$

where θ_c is the angle through which the electron is scattered in that frame. Note θ_c is distinct from the angle θ_δ that gives the direction of the velocity of the newly energized electron in the lab frame.

In the lab frame in which the newly energized electron had been stationary, $\bar{q}^\nu = \sum_\mu \Lambda_\mu^\nu \bar{q}_f^\mu$ with the sign of v_f in Lorentz transformation matrix, Eq. (25), changed. The time-like component $\bar{q}^0 = \gamma m_e c$ is given by the Lorentz transformation

$$\bar{q}^0 = \left(1 - \frac{v_f^2}{c^2}\cos\theta_c\right)\gamma_f^2 m_e c \quad (29)$$

which can be used to determine $\cos\theta_c$ in terms of γ, γ_f, and v_f/c. When this relation is used in the Lorentz transformed $\hat{x}$ component in the lab frame, $\bar{q}_x \equiv \bar{q}^1$, the result is

$$\bar{q}_x = \left(\frac{v_f}{c} - \frac{v_f}{c}\cos\theta_c\right)\gamma_f^2 m_e c = \frac{\gamma - 1}{v_f/c} m_e c. \quad (30)$$

The form of the Lorentz factor implies square of the three momentum is $\bar{q}^2 = (\gamma^2 - 1)(m_e c)^2$. The angle θ_δ between the momentum of newly energized electron and the momentum of the energetic electron is defined by $\cos^2\theta_\delta \equiv \bar{q}_x^2/\bar{q}^2$. Using Eqs. (26) and (30),

$$\cos^2\theta_\delta = \frac{\gamma_e + 1}{\gamma_e - 1}\frac{\gamma - 1}{\gamma + 1}. \quad (31)$$

2. Pitch distribution

This section will derive the pitch distribution of electrons that are energized by the collisions of an energetic electron that is moving in the $\hat{x}$ direction. When ϑ_e is the pitch angle of the fast electron relative to the magnetic field and φ_e the gyrophase, the magnetic field direction is

$$\hat{b} = \hat{x}\cos\vartheta_e - \hat{y}\sin\vartheta_e\cos\varphi_e + \hat{z}\sin\vartheta_e\sin\varphi_e. \quad (32)$$

The newly energized, or secondary, electron that results from the collision is moving in the direction

$$\hat{s} = \hat{x}\cos\theta_\delta + \hat{y}\sin\theta_\delta\cos\varphi_\delta + \hat{z}\sin\theta_\delta\sin\varphi_\delta, \quad (33)$$

where θ_δ is the angle between its momentum and that of the energetic electron.

The pitch of the newly energized secondary electron is $\lambda \equiv \hat{s}\cdot\hat{b}$, so

$$\lambda = \cos\theta_\delta\cos\vartheta_e + \sin\theta_\delta\sin\vartheta_e\cos\varphi_a, \quad (34)$$

where $\varphi_a \equiv -(\varphi_e + \varphi_\delta)$ is an arbitrary angle since it depends on the arbitrary pitch angles. Equation (31) gives $\cos\theta_\delta$, $\sin\theta_\delta = +\sqrt{1 - \cos^2\theta_\delta}$, and $\lambda_e \equiv \cos\vartheta_e$. The pitch angle of the secondary electron is then

$$\lambda = \lambda_1 + \lambda_2\cos\varphi_a, \quad (35)$$

where λ_1 is given by Eq. (16), λ_2 is given by Eq. (17), and φ_a has an equal probability of having any value in the range $2\pi \geq \varphi_a > 0$.

The probability that a newly energized electron has a pitch in a certain range is proportional to the range of φ_a

over which it can have this value of λ, and $d\varphi_a/d\lambda = -1/(\lambda_2 \sin\varphi_a)$. The pitch λ has a maximum and a minimum value since it must lie in the range $|\lambda-\lambda_1| \leq \lambda_2$. Since $\int_{\lambda_{min}}^{\lambda_{max}}(d\varphi_a/d\lambda)d\lambda = -\pi$, the probability that a new energized electron will have a pitch λ is given by Eq. (15).

IV. CONTROL OF AVALANCHE SEED

The avalanche mechanism can amplify the number of runaway electrons by $\exp(|\Delta\psi_p|/\psi_c)$. Nevertheless, a certain number of seed runaways are required to initiate an avalanche. This section considers the number of seed runaways that are required and how the high energy electrons that were part of a Maxwellian distribution before a thermal quench can be prevented from forming that seed. When the thermal quench is on a time scale longer than ~40 ms, it will be shown that pre-thermal-quench energetic electrons do not provide a seed. A similar cooling time is required to prevent mildly relativistic electrons that result from lower hybrid current drive from providing a seed that is so strong that little avalanching is required. Studies of the generation of seed electrons by faster cooling have been reported by Smith and collaborators.[12–14]

Other sources of seed electrons are problematic. The only intrinsically radioactive component in a fusion plasma is tritium, and the maximum energy of its beta-decay electrons is 18.6 keV. Alpha particles can have energies up to 3.5 MeV, but the resulting speed of a cold electron that is struck by an alpha particle is less than twice the speed of the alpha particle, which means the maximum kinetic energy is $4m_e/m_\alpha$ times the energy of the alpha particle. That is, less than 1.9 keV. When the critical energy for runaway is greater than 18.6 keV, neither provides a seed. High energy electrons from the region outside the plasma cannot cross the magnetic field between the walls and the plasma through their gyromotion when their Lorentz factor is less than about 10^3. Gamma rays from decays in the wall could produce runaways through Compton scattering though the cross section is small.

Under the assumption that background-electron drag is the only impediment to the transfer of the current from near-thermal to runaway electrons, the magnetic axis in ITER encloses sufficient toroidal flux for $\mu_0 R_{ITER} I_p/\psi_c \sim 50$ e-folds in the number of runaway electrons when $I_p = 15$ MA. If all the e-folds were used to produce the runaway current, the runaway current would of negligible magnitude. Even at 15 MA, there can be not more than about forty e-folds and still have a dangerous runaway current.

A. Required number of runaways

Number of relativistic electrons required to carry a current I_p is

$$N_r = \frac{2\pi R}{ec} I_p \approx e^{44} \frac{I_p}{15\text{MA}} \frac{R}{R_{ITER}}. \tag{36}$$

In forty-four e-folds, one runaway electron could be multiplied into a sufficient number of runaway electrons to carry the entire current in ITER. Even at 15 MA, the number of possible e-folds is less that forty-four for a dangerous runaway current, and at 10 MA the maximum number of e-folds for producing a dangerous runaway current is about twenty three, which means about $N_s > e^{21}$ seed electrons are required. The total number of electrons in ITER is $N_b \sim e^{53}$, so at 10 MA a dangerous runaway requires

$$\frac{N_s}{N_b} \approx \frac{n_s}{n_b} \equiv e^{-\sigma_s} \gtrsim e^{-32}. \tag{37}$$

B. Hot-Maxwellian as avalanche seed

The most obvious source of seed electrons for an avalanche is the remaining tail of the high temperature, T_h, Maxwellian distribution of the pre-thermal-quench plasma. When the cooling time $\tau_{cool} > \tau_c \approx 23\text{ms}/n_b$

$$n_s \approx n_b \exp\left(-\frac{m_e c^2}{T_h}\frac{\tau_{cool}}{\tau_c}\right). \tag{38}$$

For a typical ITER plasma, $T_h = 20$ keV, the ratio $m_e c^2/T_h \approx 25$, so the Maxwellian tail is not an adequate seed when $\tau_{cool} \gtrsim 40\text{ms}/n_b$. This assumes the cooled temperature of the background electrons satisfies $T_c < K_r/\sigma_s$, where K_r is the required kinetic energy for runaway, which must satisfy $K_r > (m_e c^2/2)(V_c/V_\ell)$ for $V_\ell \gg V_c$. The required cooled temperature is typically 100's of eV.

Halo current mitigation should use slow cooling. The time τ_c is sufficiently short compared to the wall time in ITER that it should be possible to both achieve both halo current mitigation, which is generally based on reducing the decay time of the plasma current below the wall time, and the elimination of the Maxwellian tail as a significant source of the electron runaway seed.

Radiative cooling times scale scales inversely with the electron density times the density of the radiating impurity, so a strong increase in n_b, which accompanies a rapid radiative collapse of the temperature, favors a strong runaway.

V. KINETIC EQUATION AND COULOMB SCATTERING

This section will consider the kinetic equation that determines the distribution function, f, for high energy electrons when two effects are present: electric field acceleration along the magnetic field lines and Coulomb collisions with a cold background plasma. The Coulomb collision operator will be given in its relativistically correct form with both the drag and the pitch-angle scattering terms.

It will be found that when the atomic number of the ions Z satisfies $(1+Z)E_c/E_{||} \gg 1$ the critical energy for runaway is greatly increased, Eq. (44), as is the required change in the poloidal flux to produce an e-fold in the number of runaway electrons, Eq. (45).

It is the atomic number of the ions Z, not their charge state, that enters Coulomb scattering when the energy of the electrons being scattered is far above the binding energy of the most tightly bound electrons in the ions. The background electron density n_b includes the bound as well as the free electrons. Nevertheless, the Coulomb logarithm, $\ln\Lambda$, which appears in E_c, is subtle when a significant fraction of the electrons are bound in high-Z impurity ions. The quantity Λ is the ratio of the largest distance over which the scattered particle feels the charge of the scatterer divided by the closest approach of the scattering particle to the scatterer, which can be due to either classical or quantum effects.[4] When the scattering particles are free electrons and ionized ions, the largest distance is the Debye length. When the scatterer is either a bound electron or a less than fully ionized ion, the largest distance is related to the size of the electron cloud of the partially ionized ion. Rosenbluth and Putvinski[2] took bound electrons as having half the effect of free electrons. Figures showing more complete calculations of the effect of bound electrons on the scattering have been presented by Aleynikov *et al*.[15]

A. Electric field acceleration

Letting $\hat{z}$ be the direction of the magnetic field, the term in the kinetic equation that gives the effect of the parallel electric field on the electron distribution function is

$$\begin{aligned} eE_{||}\hat{z}\cdot\frac{\partial f}{\partial\vec{p}} &= eE_{||}\left(\hat{z}\cdot\hat{p}\frac{\partial f}{\partial p}+\frac{1}{p}\frac{\partial f}{\partial\vartheta}\hat{z}\cdot\hat{\vartheta}\right),\\ &= eE_{||}\left(\lambda\frac{\partial f}{\partial p}+\frac{1-\lambda^2}{p}\frac{\partial f}{\partial\lambda}\right),\\ &= eE_{||}\left(\frac{\lambda}{p^2}\frac{\partial\left(p^2 f\right)}{\partial p}+\frac{\partial}{\partial\lambda}\left(\frac{1-\lambda^2}{p}f\right)\right), \end{aligned} \quad (39)$$

where $\hat{z}\cdot\hat{p}=\cos\vartheta$ and $\hat{z}\cdot\hat{\vartheta}=-\sin\vartheta$. The pitch angle of the electron relative to the magnetic field is ϑ, the pitch $\lambda\equiv\cos\vartheta$, and the magnitude of the electron momentum $p=|\vec{p}|=\gamma m_e v$.

B. Coulomb collisions

The collision operator of high energy electrons colliding with cold background electrons and ions can be written in spherical momentum space coordinates as

$$\begin{aligned} \frac{C_c(f)}{eE_c} &= \frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2\frac{c^2}{v^2}f\right)\\ &\quad +(1+Z)\frac{m_e c^2}{2vp^2}\frac{\partial}{\partial\lambda}\left((1-\lambda^2)\frac{\partial f}{\partial\lambda}\right). \end{aligned} \quad (40)$$

This operator is given in Secs. II and III of Karney and Fisch[16] and is based on the relativistic collision operator of Beliaev and Budker.[17]

C. Pitch-angle scattering and required poloidal flux

When $\gamma-1\gg 1$, the importance of pitch-angle scattering is reduced by a factor $1/\gamma$ compared to the term in the kinetic equation that gives the effect of the electric field acceleration. Pitch-angle scattering dominates $E_{||}$ acceleration when

$$\gamma<\gamma_\lambda\equiv(1+Z)\frac{E_c}{E_{||}} \quad (41)$$

assuming $\gamma_\lambda\gg 1$.

At any γ for which pitch-angle scattering dominates $E_{||}$ acceleration, the electron distribution function has a nearly isotropic pitch distribution. The parallel current requires a deviation ϵ_a from isotropy. When the current carriers are relativistic electrons,

$$j_{||}=\epsilon_a e n_r c. \quad (42)$$

The deviation from isotropy is given by the ratio of the electric-field acceleration to the pitch-angle scattering, γ/γ_λ, or

$$\epsilon_a\approx\gamma E_{||}/(1+Z)E_c \quad (43)$$

for relativistic electrons, which means with a kinetic energy $(\gamma-1)m_e c^2\gg m_e c^2$.

Runaway is only possible when electrons at a kinetic energy $(\gamma-1)m_e c^2$ obtain more power from the electric field, $E_{||}j_{||}$, where $j_{||}$ is their current density, than they lose to drag on background electrons. The drag is $en_r cE_c$ when $\gamma-1\gtrsim 1$. The implication is that the required γ factor for runaway is

$$\gamma_r-1\approx(1+Z)\left(\frac{E_c}{E_{||}}\right)^2. \quad (44)$$

The poloidal flux change required to increase the number of runaways by a factor of e is predicted by Eq. (19) to be inversely proportional to γ_r-1, so

$$\psi_{e-fold}=\sqrt{\frac{\pi}{3}}\frac{(1+Z)E_c}{E_{||}}\psi_c, \quad (45)$$

where ψ_c is the required flux change without pitch-angle scattering, Eq. (21). The $\sqrt{\pi/3}$ in the equation for ψ_{e-fold} is implied by Eq. (18) of Rosenbluth and Putvinski[2] when $1+Z\gg 1$.

Pitch-angle scattering can cause relativistic electrons to become mirror trapped in the magnetic field variation when their $\gamma\lesssim(1+Z)E_c/E_{||}$ before they can have a significant acceleration. In non-axisymmetric systems, trapped electrons can drift across the magnetic surfaces at a typical speed $\delta(\rho/R)v$ where $\rho=\gamma m_e v/B$ is their gyroradius, v their velocity, and δ is the non-axisymmetry in the magnetic field. The time they stay trapped is of order the pitch-angle scattering time τ_λ. The drift distance between collisions in ITER is then $\sim(20\gamma/n_b)\delta/(1+Z)$ in meters.

The injection of very high-Z impurities may be critical for achieving adequate runaway electron mitigation through the increase they cause in (1) the critical energy for runaway, (2) the poloidal flux change required for an avalanche e-fold, and (3) the scattering into poorly confined trapped-electron trajectories.

D. Lifshitz and Pitaevskiii anisotropy

The runaway anisotropy ϵ_a can be obtained using an electrical conductivity calculation of Lifshitz and Pitaevskiii. Lifshitz and Pitaevskii[18] derive the collision frequency for the scattering of electrons on ions of charge Z and number density n_Z

$$\nu_{ei} = \frac{4\pi Z^2 e^4 \ln\Lambda}{vp^2} n_Z. \tag{46}$$

Assuming an exponential distribution of relativistic electrons in energy, which defines a runaway temperature T_r, the average relativistic γ factor is $\bar{\gamma} = 3T_r/m_e c^2 \gg 1$. The conductivity that they derive is

$$\sigma_r = \frac{\bar{\gamma} m_e c^3}{3\pi Z e^2 \ln\Lambda} \frac{n_r}{n_e}, \tag{47}$$

where n_r/n_e is the fraction of the electrons that lie in the relativistic distribution and $Zn_Z = n_e$.

The electron current along the magnetic field is $j_{||} = \epsilon_a e n_r c$, where ϵ_a is the anisotropy of the relativistic electrons. Since $j_{||} = \sigma_r E_{||}$, the anisotropy is

$$\epsilon_a = \frac{\sigma_r E_c}{e n_r c} \frac{E_{||}}{E_c}, \tag{48}$$

$$= \frac{4}{3} \frac{\bar{\gamma}}{Z} \frac{E_{||}}{E_c} \tag{49}$$

when $\epsilon_a \ll 1$.

VI. SYNCHROTRON RADIATION

The high-energy electrons that carry a runaway current not only lose energy through drag on background electrons but also through synchrotron radiation, which arises from the electron acceleration associated with cyclotron motion. Synchrotron radiation enters the kinetic equation for the high energy electrons in the form of a collision operator, Eq. (50), which will be derived in this section.

Hazeltine and Mahajan[19] derived the effective collision operator for synchrotron radiation for sub-relativistic electrons $(\gamma - 1) \ll 1$ based on a calculation of the radiative reaction given by Rohrlich.[20] However, the power loss through synchrotron radiation, P_s, which increases as γ^2, generally competes with Coulomb drag as an energy loss mechanism only when $(\gamma - 1) \gg 1$.

Rohrlich's expression for the four force of the radiative reaction[20] can be used to obtain the synchrotron collision operator for electrons for an arbitrary Lorentz factor γ. The collision operator that represents the effects of synchrotron radiation is derived in Sec. VI A in terms of the force $\vec{f}_s$ exerted on electrons by the synchrotron radiation. Section VI C uses Rohrlich's expression for the four force of the radiative reaction to obtain $\vec{f}_s$ and the collision operator that represents the effects of synchrotron radiation. The correct form for this operator has been given in spherical momentum coordinates by Stahl *et al.*[21]

$$C_s(f) = \frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2 \frac{\gamma p(1-\lambda^2)}{\tau_s} f\right) - \frac{\partial}{\partial \lambda}\left(\frac{\lambda(1-\lambda^2)}{\gamma \tau_s} f\right). \tag{50}$$

$$\tau_s \equiv \frac{3}{2} \frac{4\pi\epsilon_0 (m_e c)^3}{e^4 B^2}, \tag{51}$$

where p is the momentum, $\lambda \equiv \cos\vartheta$ is the pitch and ϑ the pitch angle relative to the magnetic field direction of the electron. The characteristic time for synchrotron radiation is τ_s. The radiated power,

$$P_s = \frac{1}{\tau_e} \frac{p_\perp^2}{m_e}, \tag{52}$$

is implied by the first term in C_s, Eq. (62), where $p_\perp^2 = (1-\lambda^2)p^2$.

Andersson *et al.*[22] gave a related expression for the effect of synchrotron radiation on the electron kinetic equation, which has been used as a basis for a number of papers including.[21] The result by Andersson *et al.* was not originally strictly consistent with the required form, Sec. VI A, of a velocity space divergence.

The momentum-drag term in C_s is a factor of γ^2 larger than the term which exerts a drag on the pitch. Consequently, only the momentum-drag term is important when γ–$1 \gg 1$.

The drag force of synchrotron radiation is only important when it exceeds the Coulomb collisional drag on the background electrons. The ratio of the magnitude of the synchrotron radiation force to the magnitude of the collisional drag force is given by

$$\frac{P_s/c}{eE_c} \equiv \alpha_s \gamma^2 (1-\lambda^2), \tag{53}$$

$$\alpha_s = \frac{2}{3} \frac{\omega_{c0}^2}{\omega_{p0}^2} \frac{1}{\ln\Lambda} \approx 9.7 \times 10^{-2} \frac{B^2}{n_b}, \tag{54}$$

where ω_{c0}/ω_{p0} is the ratio of the cyclotron to the plasma frequency of the background electrons, B is in Tesla, and n_b is the number density of background electrons is in units of $10^{20}/\mathrm{m}^3$. Even when synchrotron drag exceeds Coulomb drag, the effects can be small when $E_{||} \gg E_c$.

Synchrotron radiation is of greatest importance when γ–$1 \gg 1$, and simplified derivations are possible in this limit. Section VI B 1 uses expressions from Landau and Lifshitz in *The Classical Theory of Fields*[23] in the γ–$1 \gg 1$ limit. Equation (52) for the radiated power can be obtained from their Eqs. (73.7) and (74.2), and the force on a charged particle due to synchrotron radiation $\vec{f}_s$ is given in their Sec. 76. Section VI B 2 derives the force $\vec{f}_s$ from the radiative power loss using the properties of the Lorentz transformation in the large γ limit.

The published literature is confusing on the synchrotron radiation of an electron aligned with a curved magnetic field line. No matter how well aligned an electron may be with a curved magnetic field, it will radiate due to its perpendicular motion. A non-zero perpendicular velocity $\vec{v}_\perp$ is required by the relativistic equation of motion, which is $\gamma m_e d\vec{v}/dt = -e\vec{v} \times \vec{B}$. Writing $\vec{v} = v_{||}\hat{b} + \vec{v}_\perp$, where $\hat{b} \equiv \vec{B}/B$, then in the limit $|\vec{v}_\perp / v_{||}| \to 0$, the acceleration term is $d\vec{v}/dt = (dv_{||}/dt)\hat{b} + v_{||}^2\vec{\kappa}$, where $\vec{\kappa} \equiv \hat{b} \cdot \vec{\nabla}\hat{b}$ is the field-line curvature with $R \equiv 1/|\vec{\kappa}|$ the radius of curvature; $\hat{b} \cdot \vec{\kappa} = 0$, which implied by $\hat{b} \cdot \hat{b} = 1$. In the $|\vec{v}_\perp / v_{||}| \to 0$ limit,

$dv_{||}/dt = 0$ and $|\gamma m_e v_{||}^2/R| = |e\vec{v}\times\vec{B}|$. This implies an inequality on the pitch-angle ϑ of the electron relative to the magnetic field

$$|\tan\vartheta| \equiv \left|\frac{\vec{v}_\perp}{v_{||}}\right| \geq \gamma\frac{m_e v_{||}}{eBR}. \quad (55)$$

A. Form of synchrotron collision operator

Since synchrotron radiation does not change the number density of electrons in ordinary space but does change their density in momentum space, the electron kinetic equation when only synchrotron radiation is retained has the form

$$\frac{\partial f}{\partial t} = C_s(f), \quad (56)$$

$$C_s(f) = -\frac{\partial}{\partial\vec{p}}\cdot\vec{\mathcal{F}}_s. \quad (57)$$

The effective collision operator for synchrotron radiation is a momentum-space divergence of a momentum-space flux $\vec{\mathcal{F}}_s$.

The drag force on electrons produced by synchrotron radiation can be obtained by multiplying Eq. (56) by the momentum $\vec{p}$ and integrating over all of momentum space,

$$\frac{\partial}{\partial t}\int\vec{p}fd^3p = \int\vec{p}C_s(f)d^3p = \int\vec{\mathcal{F}}_s d^3p, \quad (58)$$

with the use of the identity $\int\vec{x}\vec{\nabla}\cdot\vec{v}d^3x = -\int\vec{v}d^3x$, which is easily derived in ordinary space, but of course applies in momentum space as well.

The collision operator that gives the effect of synchrotron radiation on the electron distribution function is more transparent in spherical momentum coordinates. The operator $(\partial/\partial\vec{p})\cdot\vec{\mathcal{F}}_s$, where $\vec{\mathcal{F}}_s = \mathcal{F}_s\hat{p}$, can be written in spherical momentum coordinates using the standard expression for the divergence in spherical coordinates

$$\frac{\partial}{\partial\vec{p}}\cdot\vec{\mathcal{F}}_s = \frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2\hat{p}\cdot\vec{\mathcal{F}}_s\right) + \frac{1}{p\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\hat{\vartheta}\cdot\vec{\mathcal{F}}_s\right). \quad (59)$$

The distribution function $f(\vec{x},\vec{p},t)$ for an individual electron is proportional to a delta function in momentum space, so under the standard assumption that the electrons radiate incoherently Eq. (58) implies

$$\vec{\mathcal{F}}_s = \vec{f}_s f(\vec{x},\vec{p},t), \quad (60)$$

where $\vec{f}_s$ is the force on an individual electron due to synchrotron radiation.

The energy of an electron, including its rest mass energy, is $\gamma m_e c^2$, so Eq. (50) implies the power loss through synchrotron radiation is

$$\frac{\partial}{\partial t}\int\gamma m_e c^2 f d^3p = \int\gamma m_e c^2 C_s(f)d^3p, \quad (61)$$

$$= -\int P_s f d^3p \quad (62)$$

as expected using the identity $d(\gamma m_e c^2)/dp = v$, the speed of the electron.

B. Highly relativistic derivation

The expression for the synchrotron collision operator can be derived simply in the highly relativistic limit using either expressions from Landau and Lifshitz in *The Classical Theory of Fields*[23] or from properties of the Lorentz transformation in terms of the radiated power. Both derivations are given in this section.

1. Landau and Lifshitz expressions

Landau and Lifshitz in Sec. 76 of *The Classical Theory of Fields*[23] give the force exerted by synchrotron radiation on an individual, highly relativistic electron

$$\vec{f}_s = -P_s\frac{\vec{v}}{c^2}. \quad (63)$$

Using this expression for $\vec{f}_s$, Eqs. (59) and (60) give Eq. (50) for the synchrotron collision operator in the highly relativistic limit. The pitch term in the collision operator is zero because $\hat{p}\cdot\hat{\vartheta} = 0$. Note, $\hat{p}\equiv\vec{p}/|\vec{p}| = \vec{v}/|\vec{v}|$ since $\vec{p} = \gamma m_e\vec{v}$. The result is Eq. (50).

The last term in Eq. (76.3) of *The Classical Theory of Fields* has a misprint and should read $(2e^4/3m^2c^5)u^i(F_{k\ell}u^\ell)(F^{km}u_m)$. In the second paragraph below Eq. (76.3) is a clause: *those terms in the space components of the four-vector* (76.3) *increase most rapidly which come from the third derivatives of the components of the four-velocity*. This clause should read that it is the term proportional to the four-velocity cubed, which increases as γ^3, that increases most rapidly. This is the term in Eq. (76.3) that has the misprint. Despite these misprints, their expression for the force on a single electron in the strongly relativistic regime is correct. In particular, the force is opposite to the total velocity and not just its perpendicular component.

2. Synchrotron force-power relation

A Lorentz transformation into the frame of reference in which the electron is moving with velocity $\vec{v}$ from a frame in which it is instantaneously at rest demonstrates that as $\gamma\to\infty$ the synchrotron radiation force (1) is aligned with the total velocity of the electron and (2) is uniquely determined by the synchrotron power loss P_s.

The relativistic equation of motion is $dp^\mu/d\tau = K^\mu$, where $d\tau = dt/\gamma$ is time interval in the frame of rest of the particle and K^μ is the four force. The ordinary force $\vec{f}$ on a particle is given by the three space-like components of the four force $K^{1,2,3}$, divided by γ. The power transferred to the particle is given by cK^0/γ. Since $\sum_\mu p_\mu p^\mu = -(m_e c)^2$, the four force must obey the constraint $\sum_\mu p_\mu K^\mu = 0$.

Let $\mathcal{K}^\mu$ be the four force on an electron in the frame in which it is at rest and K^μ be the four force in the frame in which it is moving. The relation between the four force in the two frames is $K^\nu = \sum_\mu \Lambda^\nu_\mu \mathcal{K}^\mu$. The convention of Λ^ν_μ of Eq. (25) implies the velocity difference between the two frames is $\vec{v}_f = -\vec{v}$ with the $\hat{x}$ axis, which has index 1, aligned with the velocity $\vec{v}$. The Lorentz transformation implies $K^0 = \gamma(\mathcal{K}^0 + \mathcal{K}^1 v/c)$, $K^1 = \gamma(\mathcal{K}^0 v/c + \mathcal{K}^1) = K^0$, $K^2 = \mathcal{K}^2$, and $K^3 = \mathcal{K}^3$. Since the usual three-space force, $\vec{f}$, is the space-like components of the four-force divided by γ, the space-like force components that are perpendicular to the particle velocity scale as $1/\gamma$ compared either to the space-like component aligned with the velocity, K^1/γ, which is equal to K^0/γ, or to the power transfer $cK^0/\gamma = -P_s$.

The constraint $\sum_\mu p_\mu K^\mu = 0$, which in the frame in which the electron is at rest is $m_e c\mathcal{K}^0 = 0$, implies $\mathcal{K}^0 = 0$. Unlike $\mathcal{K}^0$, the space-like components of the four force $\mathcal{K}^{1,2,3}$ need not vanish.

C. General derivation

Equation (4) in Rohrlich's paper[20] gives the four forces due to radiation for a particle moving in given electric and magnetic fields. In the rest frame of the particle,

$$\mathcal{K}^\mu = \frac{2}{3}\frac{e^4}{4\pi\epsilon_0 m_e^2 c^2}\sum_\alpha \left(F^{\mu\alpha}F_{\alpha 0} - \delta^\mu_0 F^{0\alpha}F_{\alpha 0}\right), \tag{64}$$

where the electromagnetic field tensor is

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}. \tag{65}$$

Writing the four force as $\mathcal{K}^\mu = (K^0, \vec{\mathcal{K}})$, one finds $K^0 = 0$, but

$$\vec{\mathcal{K}} = \frac{2}{3}\frac{e^4}{4\pi\epsilon_0 m_e^2 c^2}\vec{E}^{(r)} \times \vec{B}^{(r)}, \tag{66}$$

where the superscript (r) has been added to the electric and magnetic fields in which the particle is moving to indicate they are to be determined in the rest frame of the particle.

In the lab frame, the particle has a velocity $v\hat{x}$ and is moving in a magnetic field $\vec{B}$ with the electric field zero, $\vec{E} = 0$. A Lorentz transformation of the electric field, which follows from a Lorentz transformation of the electromagnetic field tensor, gives $E_x^{(r)} = 0$, $E_y^{(r)} = -\gamma v B_z$, and $E_z^{(r)} = \gamma v B_y$. The transformation of the magnetic field gives $B_x^{(r)} = B_x$, $B_y^{(r)} = \gamma B_y$, and $B_z^{(r)} = \gamma B_z$. Therefore, the four forces in the rest frame of a particle that is moving with a velocity $v\hat{x}$ are

$$\mathcal{K}^\mu = k_s v\left(0, -\gamma^2\frac{B_y^2 + B_z^2}{B^2}, \gamma\frac{B_x B_y}{B^2}, \gamma\frac{B_x B_z}{B^2}\right), \tag{67}$$

$$k_s \equiv \frac{2e^4 B^2}{3(4\pi\epsilon_0)m_e^2 c^3}. \tag{68}$$

Note that $\vec{B}$ is the magnetic field in which the particle is moving, not the radiation field. For strongly relativistic electrons, the γ^2 term in the four forces is dominant and gives the same result as used in Sec. VI B.

The three spatial components of the four force in the laboratory frame are related to the ordinary three force by $\vec{f}_s = \vec{K}/\gamma$, so using the identities $\vec{B}_\perp = \vec{B} - \vec{v}\cdot\vec{B}/v$ and $\lambda = (\vec{v}\cdot\vec{B})/(vB)$,

$$\frac{\vec{f}_s}{k_s} = -\gamma^2\frac{B_\perp^2}{B^2}\vec{v} + \frac{\vec{v}\cdot\vec{B}\vec{B}_\perp}{B^2}, \tag{69}$$

$$= -\gamma^2\vec{v} + \frac{(\vec{v}\cdot\vec{B})\vec{B}}{B^2} + \left(\gamma^2 - 1\right)\frac{(\vec{v}\cdot\vec{B})^2}{(vB)^2}\vec{v}, \tag{70}$$

$$= -\gamma^2(1-\lambda^2)\vec{v} - \lambda\left(\lambda\vec{v} - v\frac{\vec{B}}{B}\right). \tag{71}$$

As $\gamma \to 1$, the force is $\vec{f}_s = -k_s\vec{v}_\perp$, which agrees with Ref. 19. The two important components of the synchrotron force for scattering are

$$\vec{f}_s\cdot\hat{p} = -k\gamma^2(1-\lambda^2)v = -k\frac{p_\perp^2}{m_e^2 v}, \tag{72}$$

$$\vec{f}_s\cdot\hat{\vartheta} = -kv\cos\vartheta\sin\vartheta. \tag{73}$$

Since $\vec{v}\cdot\hat{p} = v$, $\vec{v}\cdot\hat{\vartheta} = 0$, $\vec{B}\cdot\hat{p} = \lambda B$, and $\vec{B}\cdot\hat{\vartheta} = -\sin\vartheta B$. These equations together with Eqs. (59) and (60) give Eq. (50) for the synchrotron collision operator.

VII. POWER BALANCE FOR RUNAWAYS

The energy equation is the $(\gamma-1)m_e c^2$ moment of the kinetic equation, which is calculated by multiplying the kinetic equation for the high energy electrons,

$$\frac{\partial f}{\partial t} - eE_{||}\hat{z}\cdot\frac{\partial f}{\partial\vec{p}} = C_c(f) + C_s(f) + S, \tag{74}$$

by the kinetic energy of a runaway electron $(\gamma-1)m_e c^2$ and integrating over $d^3p = 2\pi p^2 dp d\lambda$. The average relativistic γ factor, or equivalently average energy, is

$$\bar{\gamma} \equiv 1 + \frac{\int(\gamma-1)fd^3p}{\int fd^3p}. \tag{75}$$

The number of runaway electrons per unit volume is $n_r = \int fd^3p$.

The power balance equation will be derived using the simplified source function S of Rosenbluth and Putvinski, Sec. III A, of the avalanche mechanism. Their S adds particles but otherwise plays a negligible role in power balance because the critical relativistic factor for runaway, γ_r, satisfies $\gamma_r - 1 \ll \bar{\gamma} - 1$. The avalanche source function is the only term in the kinetic equation that does not conserve particles.

The term in the kinetic equation involving the parallel electric field, $\int(\gamma-1)m_ec^2eE_{||}\hat{z}\cdot(\partial f/\partial\vec{p})d^3p$, can be analyzed using Eq. (39) and the identity $\partial\gamma/\partial p = v/m_ec^2$, which follows from writing the Lorentz factor as $\gamma=\sqrt{1+(p/m_ec)^2}$. The resulting expression is $E_{||}j_{||}$ where the parallel current $j_{||}\equiv -e\int\lambda vfd^3p$.

The equation $\partial\gamma/\partial p = v/m_ec^2$ can be also used to simplify the term $\int(\gamma-1)m_ec^2C_cd^3p$. The resulting expression is $-eE_c\int(c^2/v)fd^3p$. When $\bar{\gamma}-1\gg1$, this expression simplifies to $-E_cecn_r$.

A. Without synchrotron losses

Ignoring the power loss from synchrotron radiation, the $(\gamma-1)m_ec^2$ moment of Eq. (74) implies

$$\frac{d\bar{\gamma}}{dt}m_ec^2n_r+(\bar{\gamma}-1)m_ec^2\frac{dn_r}{dt}=E_{||}j_{||}-E_cecn_r. \qquad (76)$$

The rate of change of the number of runaways, dn_r/dt, is given by Eq. (19). Using $\tau_c\equiv(eE_c/m_ec)$, Eq. (6)

$$\tau_c\frac{d\bar{\gamma}}{dt}+\frac{\bar{\gamma}-1}{2(\gamma_r-1)\ln\Lambda}=\frac{E_{||}}{E_c}\frac{j_{||}}{en_rc}-1. \qquad (77)$$

When $E_{||}$ is time independent, Eq. (77) implies $\bar{\gamma}$ relaxes to a steady state value on the time scale τ_R

$$\bar{\gamma}(t)-1=(\bar{\gamma}(\infty)-1)(1-e^{-t/\tau_R}), \qquad (78)$$

$$\tau_R\equiv2(\gamma_r-1)\tau_c\ln\Lambda, \qquad (79)$$

$$\frac{\bar{\gamma}(\infty)-1}{2(\gamma_r-1)\ln\Lambda}=\left(\frac{E_{||}}{E_c}\frac{j_{||}}{en_rc}-1\right). \qquad (80)$$

Equation (19) for dn_r/dt can be rewritten using Eq. (80) as

$$\frac{d\ln n_r}{dt}=\frac{e}{m_ec}\frac{E_{||}(j_{||}/en_rc)-E_c}{\bar{\gamma}(\infty)-1}. \qquad (81)$$

The rate at which the runaways exponentiate is not affected by whether t is long or short compared to τ_R, provided $\bar{\gamma}(t)\gg1$, but the average kinetic energy of the runaways increases linearly in time, $\bar{\gamma}(t)-1=(t/\tau_r)(\bar{\gamma}(\infty)-1)$ when $t\ll\tau_R$. To simplify notation, $\bar{\gamma}$ will denote $\bar{\gamma}(\infty)$.

The necessary condition for runaway production, $E_{||}>E_cen_rc/j_{||}$, only approximates the sufficient condition when the runaway electrons are moving in close alignment with the magnetic field, so $j_{||}\approx en_rc$. Pitch-angle scattering tends to isotropize the electrons in pitch, which can reduce the current density of runaway electrons from its maximum value en_{rc}.

In general, the current density of runaway electrons is $j_{||}=\epsilon_aen_rc$, where ϵ_a is the anisotropy of the runaway electrons, $\epsilon_a\le1$. The critical electric field for runaway is $E_{||}>E_c/\epsilon_a$. When $E_{||}\gg E_c/\epsilon_a$, Eq. (80) implies

$$\frac{\bar{\gamma}-1}{\gamma_r-1}=2\epsilon_a\ln\Lambda\frac{E_{||}}{E_c}. \qquad (82)$$

Equation (82) has Eq. (22) as a special case, $\bar{\gamma}-1\approx2\ln\Lambda$, when $\gamma_r-1=E_c/E_{||}$ and $\epsilon_a\approx1$.

With the inclusion of pitch-angle scattering, the runaway condition becomes $\gamma_r-1=(1+Z)(E_c/E_{||})^2$, Eq. (44). Equation (82) implies $\bar{\gamma}-1=2(1+Z)(E_c/E_{||})\epsilon_a\ln\Lambda$. The implication is that pitch-angle scattering is not important for the typical runaway electron. To prove this first assume the $\epsilon_a\approx1$; the result is $\bar{\gamma}\gg(1+Z)E_c/E_{||}$, so pitch-angle scattering is not important. Then assume $\epsilon_a\approx\bar{\gamma}E_{||}/(1+Z)E_c\lesssim1$. The equation $\bar{\gamma}-1=2(1+Z)(E_c/E_{||})\epsilon_a\ln\Lambda$ implies $1=2\ln\Lambda$, which is a contradiction. Consequently when $(1+Z)E_c/E_{||}>1$, pitch-angle scattering is negligible for the typical runaway electron, $\epsilon_a\approx1$, and

$$\bar{\gamma}-1\approx2\ln\Lambda\frac{(1+Z)E_c}{E_{||}}, \qquad (83)$$

where $E_{||}$ is assumed to satisfy $(1+Z)E_c\gg E_{||}\gg E_c$.

B. Synchrotron radiation when $Z\gg1$

Section VII A showed that even when $(1+Z)E_c/E_{||}\gg1$, the typical value for the runaway kinetic energy $(\bar{\gamma}-1)m_ec^2$ is sufficiently high that pitch-angle scattering has a small effect for most runaway electrons. The effects of synchrotron emission are then limited because with only Coulomb effects giving pitch-angle scattering the width in pitch angle can narrow to $\delta\vartheta\approx\gamma_\lambda/\bar{\gamma}=(1+Z)(E_c/E_{||})/\bar{\gamma}$. Using Eq. (83)

$$\delta\vartheta\approx\frac{1}{2\ln\Lambda}. \qquad (84)$$

This expression for $\delta\vartheta$ follows from properties of pitch angle scattering.[24] A particle that initially has a pitch λ_0 can have a range of values of pitch λ_τ after a time τ. Sampling from a binomial distribution and assuming the collision frequency ν times τ is small, $\lambda_\tau=(1-\lambda_0)(1-\nu\tau)\pm\sqrt{(1-\lambda_0^2)\nu\tau}$, where $\pm$ is a random sign. The largest magnitude of λ_τ is $|\lambda_\tau|=1-\nu\tau/2$.

Assuming synchrotron radiation is more important than drag, the ratio the power gained from the electric field, $E_{||}en_rc$ to that lost by synchrotron radiation $P_sn_r=(E_cen_rc)\alpha_s\bar{\gamma}^2\delta\vartheta^2$, which implies runaway is only possible when

$$\frac{E_{||}}{E_c}>\left((1+Z)^2\alpha_s\right)^{1/3}. \qquad (85)$$

VIII. OHM'S LAW FOR TEARING MODES

Even when the current is carried by runaways, the deviation in the plasma response from an ideal Ohm's law during a resistive instability is determined by the Ohm's law of the thermal electrons.[3] Since runaway electrons are moving at the velocity of light, the only way to change the runaway current is to change the number of runaways, which is over a time scale $\tau_cE_c/(E_{||}-E_c)$. There is a long lag time between a change in the electric field and the response of the runaway current. Thermal electrons respond as $\eta\vec{j}_{th}=\vec{E}+\vec{v}\times\vec{B}$ on the time scale of their collisions. So deviations from an ideal response, such as the opening of islands, are determined by the thermal electrons.

IX. MICROTURBULENCE

The microinstability that is thought to be of greatest potential importance to the runaway issue on ITER is an instability of the whistler waves.[25,26] This instability requires a strong anisotropy of the perpendicular to parallel electron momentum in a beam $|p_\perp/p_{||}| \ll 1$, and is stabilized quasi-linearly by spreading $p_\perp$. The $p_\perp$ spreading has two effects: (1) The power loss from synchrotron radiation, which depends quadratically on $p_\perp$ is enhanced.[27] (2) If pitch-angle scattering were large enough to reverse the direction of electrons along the magnetic field, which means $p_{||}$ reversal, then the effect on runaways is similar to pitch-angle scattering when $\gamma < (1+Z)E_c/E_{||}$. Unfortunately, quasi-linear stabilization appears to occur well before the electron momentum along the magnetic field is reversed.[26]

X. DISCUSSION OF PHYSICS AND MITIGATION

The achievement of the ITER mission will require the successful avoidance of large currents of relativistic electrons. The potential for transferring the plasma current from thermal to relativistic electrons exists whenever the poloidal magnetic flux content of an ITER plasma changes on a time scale faster than tens of seconds. The danger to the ITER device posed by a large current of relativistic electrons requires a focus on avoidance rather than on the control of such currents if they occur.

The poloidal flux must change faster than a critical value for a significant number of relativistic electrons to persist in an ITER plasma. This time scale is discussed in Sec. II and can be expressed as a critical loop voltage, Eq. (3), which is $V_c \approx 3n_b$ Volts when pitch angle scattering is ignored; n_b is the background electron density in units of $10^{20}/\mathrm{m}^3$. In many ITER operational scenarios, $n_b \approx 1$. The loop voltage times the time gives the change in the poloidal magnetic flux.

When the loop voltage is large compared to the critical loop voltage, the number of electrons above a certain energy, called the effective runaway energy K_r, will increase exponentially and naturally give a distribution of relativistic electrons with an average kinetic energy $(\bar{\gamma}-1)m_ec^2$. Ignoring pitch-angle scattering, $\bar{\gamma}-1 \approx 2\ln\Lambda$ with $\ln\Lambda$ the Coulomb logarithm. Given an initial number density of electrons with an energy above K_r, called seed electrons, the number of relativistic electrons will be $n_r = n_s \exp(|\Delta\psi_p|/\psi_{e-fold})$, where $|\Delta\psi_p|$ is the change in the poloidal flux that has occurred and ψ_{e-fold} is the flux change required for an e-fold. Ignoring pitch-angle scattering, $\psi_{e-fold} = \psi_c \approx 2.3\,\mathrm{V\cdot s}$. This increase in the number of relativistic electrons continues until either all of the available poloidal flux has disappeared or all of the plasma current is carried by relativistic electrons. The time scale for the current transfer, $|\Delta\psi_p|/V_\ell$, is determined by the Ohmic loop voltage, V_ℓ of the background plasma. This current transfer process, called the electron avalanche, is discussed in Sec. III and the angular distribution of the newly energized electrons is derived as a function of their kinetic energy K_s. Knowledge of this angular distribution is critical for determining the effective energy required for runaway since trapped electrons cannot obtain significant energy from the parallel electric field.

The poloidal flux change required for a current transfer can be increased by reducing the number of seed electrons. The required flux change is discussed in Sec. IV as are the benefits of slowing the plasma cooling to a time scale of $\sim$40 ms, when possible, to eliminate the high energy electrons that survive from the pre-thermal-quench Maxwellian or from current drive. Preemptive cooling of disruption prone plasmas is the only obvious method of avoiding the rapidly cooling of a naturally arising thermal quench. A 40 ms cooling time should be adequate to prevent the damaging halo currents that arise when loss of axisymmetric control results in the plasma being pushed into the wall, so the halo-current mitigation system should be made consistent with this time scale.

In addition to slowing on background electrons, the pitch angle of high energy electrons is scattered through collisions with the background plasma at a rate proportional to $1+Z$, where Z is the atomic number of the background ions. Because of the high energy of runaway electrons, whether the ions are fully ionized or not makes only a modest difference, $\sim$2, in the scattering. As discussed in Sec. V, pitch angle scattering produces a major modification of the runaway process when the parallel electric field $E_{||} \lesssim (1+Z) E_c$ or equivalently when the loop voltage $V_\ell \lesssim (1+Z)V_c$. The primary changes are: (1) The critical kinetic energy for runaway, $(\gamma_r-1)m_ec^2$, increases from $\gamma_r - 1 \approx E_c/E_{||}$ to $\gamma_r - 1 \approx (1+Z)(E_c/E_{||})^2$. (2) The poloidal flux change required for an e-fold in the number of relativistic electrons increases from $\psi_c \approx 2.3\,\mathrm{V\cdot s}$ to $\psi_{e-fold} \approx (1+Z)(E_c/E_{||})\psi_c$. (3) When the plasma is strongly non-axisymmetric, a sufficient number of high energy electrons may drift out of the plasma as trapped particles to prevent an avalanche.

Synchrotron radiation, Sec. VI, drains energy from relativistic electrons, but its importance to the process of the transfer of current from thermal to relativistic electrons appears marginal except when the background plasma density is low and the magnetic field strength is high.[27] Nevertheless, the derivation of the collision operator that gives the effect of synchrotron radiation on the electron distribution function is derived in some detail. Contrary to some expectations, the force exerted by synchrotron radiation is not aligned with the electron velocity perpendicular to the magnetic field $\vec{v}_\perp$ but with the full velocity $\vec{v}$, and this affects the form of the collision operator.

The energy moment of the kinetic equation is used in Sec. VII to find the average kinetic energy of the relativistic electrons that carry a runaway current, $(\bar{\gamma}-1)m_ec^2$. Without pitch angle scattering $\bar{\gamma}-1 \approx 2\ln\Lambda$, where $\ln\Lambda$ is the Coulomb logarithm. When $E_{||} \lesssim (1+Z)E_c$ the average energy of runaways is increased to $\bar{\gamma}-1 \approx 2(1+Z)(E_c/E_{||})\ln\Lambda$. This section also shows that it is possible for synchrotron radiation to prevent runaway in a high-Z plasma, but the range of $E_{||}/E_c$ over which this can occurs appears very limited in ITER scenarios.

Sections VIII and IX mention results that have already appeared in the literature: When tearing modes arise, the break up of magnetic surfaces occurs on the resistive time scale of the cold background plasma.[3] It appears that microturbulence does not have a major role in the phenomenon of current transfer from thermal to runaway electrons. Anomalies have been seen in experiments involving runaways,[28] which could make avoidance of a current transfer much easier. More analysis is required to know whether these empirical anomalies might be due to other effects, such as pitch angle scattering by impurities or the time scale of the experiments. Monte Carlo simulations of the kinetic equation for high energy electrons could relatively simply address much of the uncertainty.

Appendix A gives the response of the trajectory and the energy of an electron to a slippage of the poloidal relative to the toroidal magnetic flux. The rate of this slippage is the loop voltage.

A final decision will need to be made on the mitigation system that will be installed on ITER in just a few years. It is critical that simulations address the effectiveness of proposed mitigation strategies for avoiding strong currents of relativistic electrons. Fortunately, these could be done relatively easily and quickly by representing the collision operators by their Monte Carlo equivalents.[24] Simulations of what happens if the transfer of current from thermal to relativistic electrons actually occurs are far more difficult. The current carried by the relativistic electrons must be known at each point in the plasma. Fortunately, constraints on the relativistic current density make a point by point calculation of the relativistic current simpler than might be expected, Appendix B.

The issue of current transfer from thermal to relativistic electrons is central to the success of the ITER program but far less central to the success of magnetic fusion energy. For example, little change would occur in stellarator reactor designs if the constraint were imposed that the net plasma current must be smaller than 5 MA.

Even tokamaks, such as ITER, operating well above 5 MA might be designed to prevent a current transfer from thermal to runaway electrons. The poloidal field due to the net plasma current penetrates the walls surrounding the plasma on a time scale of 100's of milliseconds in ITER. When the current in the plasma collapses on a faster time scale, a toroidal current equal to the plasma current must be induced in the walls. If the wall structures were designed to minimize the forces on the walls due to this current, a minimization of $|\vec{j}\times\vec{B}|/|\vec{j}\cdot\vec{B}|$, a sudden drop in the plasma current would cause a destruction of the magnetic surfaces in the plasma and prevent a transfer of the remaining plasma current from thermal to relativistic electrons.[7,8] During normal ITER operations, the net toroidal current flowing in the walls is small, and the effect on the magnetic surfaces would be negligible. Since the natural path of currents through the wall is known, small error-field effects could be compensated with error-field control coils, but this does not appear to be required. The usual design of allowing these induced currents to flow toroidally not only eliminates their drive for the opening of the magnetic surfaces but also makes their force larger.

The danger is too great to use an empirical validation of the runaway mitigation system on ITER itself. Computational simulations are required, and this paper gives the physics that could be used to relatively easily and quickly carry out such simulations.

ACKNOWLEDGMENTS

This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Fusion Energy Sciences under Award No. De-FG02-03ER54696. The author would like to thank Igor Kaganovich and Edward Startsev for useful discussions and Per Helander for making a number of comments and suggestions.

APPENDIX A: MAGNETIC FLUX CHANGES DURING RUNAWAY ACCELERATION

Relativistic electrons have a small gyroradius ρ_e compared to characteristic plasma dimensions in ITER; $\rho_e=\gamma mc/eB\approx 1.7$ mm (γ/B), where B is in Tesla and the electron kinetic energy is $(\gamma-1)m_ec^2$. Consequently, the electron motion can be followed using the relativistic drift Hamiltonian,[29] $dp_\varphi/dt=-\partial H/\partial\varphi$, $dp_\theta/dt=-\partial H/\partial\theta$. The relativistic canonical momenta of the electron drift Hamiltonian are $p_\varphi=(\mu_0 G/2\pi B)\gamma m_e v_{||}+e\psi_p/2\pi$, and $p_\theta=(\mu_0 I/2\pi B)\gamma m_e v_{||}-e\psi_t/2\pi$. The toroidal current I and the toroidal magnetic flux ψ_t lie in the region enclosed by a magnetic surface. The poloidal current G and the poloidal magnetic flux $-\psi_p$ lie outside the same magnetic surface. When magnetic surfaces exist, the poloidal θ and the toroidal φ angles can be chosen so the magnetic field can be written in two forms[4]

$$2\pi\vec{B}=\vec{\nabla}\varphi\times\vec{\nabla}\psi_p(\psi_t,t)+\vec{\nabla}\psi_t\times\vec{\nabla}\theta, \tag{A1}$$

$$=\mu_0 G(\psi_t,t)\vec{\nabla}\varphi+\mu_0 I(\psi_t,t)\vec{\nabla}\theta + \beta_*(\psi_t,\theta,\varphi,t)\vec{\nabla}\psi_t. \tag{A2}$$

For passing electrons, one can average dp_θ/dt over θ and dp_φ/dt over φ. The averaged p_θ and p_φ are independent of time. The conservation of p_θ and p_φ in time-independent magnetic fields can be obtained from the expression for the guiding-center velocity $\vec{v}_{gc}=(v_{||}/B)\vec{\nabla}\times(\vec{A}+\rho_{||}\vec{B})$, where $\rho_{||}$ is the gyroradius calculated using the component of the velocity that is parallel to the magnetic field.[4] Equation (A1) implies $2\pi\vec{A}=\psi_t\vec{\nabla}\theta-\psi_p\vec{\nabla}\varphi$, so $\vec{A}_*\equiv\vec{A}+\rho_{||}\vec{B}$ can be written as

$$2\pi(-e)\vec{A}_*=p_\theta\vec{\nabla}\theta-p_\varphi\vec{\nabla}\varphi. \tag{A3}$$

Passing particle trajectories lies in the surfaces of the effective magnetic field $\vec{B}_*\equiv\vec{\nabla}\times\vec{A}_*$ when such surfaces exist. Since the surfaces of $\vec{B}_*$ are perturbed from those of $\vec{B}$ only by terms of order gyroradius to system size, p_θ and p_φ vary by terms only of that order. The same result does not hold for trapped particles because $\vec{B}_*$ is singular at turning points, the points where $v_{||}=0$.

Since p_θ and p_φ are conserved

$$\frac{d2\pi p_\theta/e}{dt} = \frac{\mu_0 G}{eB}\frac{d\gamma m_e v_{||}}{dt} - \frac{d\psi_p}{dt} = 0, \tag{A4}$$

$$\frac{d2\pi p_\theta/e}{dt} = \frac{\mu_0 I}{eB}\frac{d\gamma m_e v_{||}}{dt} + \frac{d\psi_t}{dt} = 0. \tag{A5}$$

Ignoring the terms that become negligible as the parallel gyroradius $\rho_{||}$ goes to zero relative to the system size, which means terms proportional to $\gamma v_{||}$ instead of its time derivative, these two equations imply

$$\frac{d\psi_p}{dt} = \frac{G}{G+\iota I}V_\ell, \tag{A6}$$

$$\frac{d\psi_t}{dt} = -\frac{I}{G+\iota I}V_\ell, \tag{A7}$$

$$\frac{d\gamma m_e v_{||}}{dt} = -\frac{eB}{\mu_0(G+\iota I)}V_\ell, \tag{A8}$$

where the rotational transform $\iota \equiv (\partial\psi_p/\partial\psi_t)_t = 1/q$ and the loop voltage $V_\ell \equiv (\partial\psi_p/\partial t)_{\psi_t}$. The total time derivatives mean along the trajectory of an electron.

The form of the poloidal flux ψ_p is non-intuitive to many. Its form can be clarified by considering cylindrical coordinates $(r,\theta,z=R\varphi)$, which means the cylinder is periodic in the z direction with a period $2\pi R$ and $\vec\nabla\varphi = \hat z/R$. Equation (A1) implies $\partial\psi_p(r,t)/\partial r = 2\pi R B_\theta$. When the poloidal flux is held fixed on a cylindrical shell at $r=b$, let $\psi_p(b,t)=0$. The poloidal flux is then $\psi_p(r,t) = 2\pi R\int_b^r B_\theta dr$. When the current density is constant for $r<a$ and zero for $a<r<b$, then for $r<a$

$$\psi_p = -\mu_0 R\left(\ln\left(\frac{b}{a}\right) + \frac{1}{2}\left(1-\frac{r^2}{a^2}\right)\right)I_p, \tag{A9}$$

where I_p is the current in the plasma. The poloidal flux enclosed by the magnetic axis, $r=0$, is negative $\psi_p(0,t) = -\mu_0 R(\ln(b/a)+1/2)I_p$ while the loop voltage, which is due to the plasma resistivity, is positive. Consequently, the poloidal flux at the axis rises towards the value of zero, which is its value at $r=b$ throughout the evolution. The toroidal flux in the cylindrical model is $\psi_t = \pi B_\varphi r^2$.

The enclosed poloidal flux depends on the current profile. When the initial current density $j(r) \propto 1 - r^2/a^2$, where a is the plasma radius, the current $I(r) = I_a(2-r^2/a^2)r^2/a^2$, the poloidal flux enclosed by the magnetic axis

$$-\psi_p = \mu_0 R\left(\frac{3}{4} + \ln\left(\frac{b}{a}\right)\right)I_a. \tag{A10}$$

In standard tokamaks, $I/G \ll 1$, and the inward pinching in toroidal flux of an accelerating electron, Eq. (A7), is small. In a cylindrical model $I/G = (B_\theta r)/(B_\varphi R) = (r/R)^2\iota(r)$, where the rotational transform $\iota = 1/q = RB_\theta/aB_\varphi$.

Equation (A1) can be generalized[4] to represent and arbitrary magnetic field for which $\vec B\cdot\vec\nabla\varphi$ is non-zero by letting ψ_p be a function of $(\psi_t,\theta,\varphi,t)$. The Appendix to Ref. 4 shows the time dependence of the vector potential is

$$\left(\frac{\partial\vec A}{\partial t}\right)_{\vec x} = -V_\ell\vec\nabla\frac{\varphi}{2\pi} + \vec u\times\vec B + \vec\nabla s, \tag{A11}$$

where the loop voltage is $V_\ell \equiv \partial\psi_p(\psi_t,\theta,\varphi,t)/\partial t$, the velocity of a (ψ_t,θ,φ) point through space is $\vec u$, and s is a well behaved function of position and time.

Since $\vec E = -\partial\vec A/\partial t - \vec\nabla\Phi$, the average loop voltage in a region covered by a magnetic field line is

$$\bar V_\ell = \frac{2\pi}{\int d\psi_t d\theta d\varphi}\int\frac{\vec E\cdot\vec B}{\vec B\cdot\vec\nabla\varphi}d\psi_t d\theta d\varphi. \tag{A12}$$

When magnetic surfaces exist the integrals are only over an infinitesimal range of ψ_t, and the averaged loop voltage can be written $V_\ell(\psi_t,t)$. Equations (A1) and (A2) then imply $1/(2\pi\vec B\cdot\vec\nabla\varphi) = \mu_0(G+\iota I)/(2\pi B)^2$, so

$$V_\ell = \mu_0(G+\iota I)\oint\frac{E_{||}}{B}\frac{d\theta d\varphi}{(2\pi)^2}. \tag{A13}$$

With the standard Ohm's law and the resistivity η constant within a surface, $\oint(E_{||}/B)d\theta d\varphi/(2\pi)^2 = \eta\langle j_{net}\rangle/\langle B\rangle$, where j_{net} is the net current density, which has the property that j_{net}/B is constant over the surface, so the averages over j_{net} and B can have any desired form. Consequently, one can write

$$\langle E_{||}\rangle = \frac{\langle B\rangle}{\mu_0(G+\iota I)}V_\ell. \tag{A14}$$

The implication is that the rate of increase of the electron momentum, Eq. (A8), is due to the surface-averaged parallel electric field as one would expect.

APPENDIX B: SIMULATION OF RUNAWAY CURRENT

The evolution of the runaway current can be followed using Monte Carlo methods. A collision operator can be converted into a Monte Carlo operator.[24] This Appendix shows that the current obtained from a Monte Carlo calculation can have far less statistical noise than might be expected because of physics constraints that imply spatially averaged quantities accurately define the local current density.

The current density of runaway electrons is $\vec j = K_n\vec B/\mu_0 + \delta\vec j$, where $\delta\vec j$ is proportional to and given by the pressure tensor of the runaway electrons. The net parallel current of the runaways, which is given by K_n, must be constant along each magnetic field line. That is, $K_n = \mu_0\langle j_{||}/B\rangle$, which means an average along the magnetic field line. By explicitly evaluating the field-line average, the noise in computing the runaway current density can be greatly reduced. The net parallel current density of the runaways is approximately a hundred times larger than their perpendicular current density.

The electric field exerts a far smaller force on runaway electrons than does the magnetic field, so force balance across the magnetic field for runaway electrons is given by

$$\vec\nabla\cdot\overleftrightarrow{P} = \vec j\times\vec B, \tag{B1}$$

where $\overleftrightarrow{P}$ is the pressure tensor of the runaway electrons and $\vec j$ is their current density.

The divergence of the runaway current obeys

$$\vec{\nabla}\cdot\vec{j} = e\frac{\partial n_r}{\partial t}, \tag{B2}$$

where $\partial n_r/\partial t$ is the rate runaways are created per unit volume, which is approximately four orders of magnitude smaller than the characteristic value of the divergence of the runaway current $|\vec{\nabla}\cdot\vec{j}| \sim en_rc/R$, where R is the spatial scale of the plasma. Therefore, one can assume

$$\vec{\nabla}\cdot\vec{j} = 0. \tag{B3}$$

Equations (B1) and (B3) imply the runaway current density can be accurately given as

$$\vec{j} = \frac{j_{||}}{B}\vec{B} + \vec{j}_\perp, \tag{B4}$$

$$\vec{j}_\perp = \frac{\vec{B}\times\vec{\nabla}\cdot\overleftrightarrow{P}}{B^2}, \tag{B5}$$

$$\vec{B}\cdot\vec{\nabla}\frac{j_{||}}{B} = -\vec{\nabla}\cdot\vec{j}_\perp. \tag{B6}$$

An average of $\vec{\nabla}\cdot\vec{j}_\perp$ over the volume of an infinitely long magnetic flux tube must vanish when the current density along the tube remains finite, so

$$\langle\vec{\nabla}\cdot\vec{j}_\perp\rangle \equiv \lim_{L\to\infty}\frac{\int_0^L \vec{\nabla}\cdot\vec{j}_\perp \frac{d\ell}{B}}{\int_0^L \frac{d\ell}{B}} = 0, \tag{B7}$$

where $d\ell$ is the differential distance along a magnetic field line.

The current density of the runaways parallel to the magnetic field is, therefore, given by

$$\mu_0\frac{j_{||}}{B} = K_n + K_{ps}, \tag{B8}$$

$$K_n \equiv \left\langle \mu_0\frac{j_{||}}{B}\right\rangle, \tag{B9}$$

$$\frac{\partial}{\partial\ell}K_{ps} = -\frac{1}{B}\vec{\nabla}\cdot\vec{j}_\perp. \tag{B10}$$

By far the largest term in the runaway current density is the net current K_n. The Pfirsch-Schlüter current, which is given by K_{ps}, and the perpendicular current density are far smaller.

The ratio of the perpendicular current to the net current is

$$\left|\frac{j_\perp}{j_{||}}\right| \approx \left|\frac{\gamma m_e n_r c^2/aB}{en_rc}\right| = \frac{\rho_r}{a}, \tag{B11}$$

where a is the plasma radius and the gyroradius of a runaway electron is $\rho_r \equiv \gamma c/\omega_{ce} = 1.7\times10^{-3}\gamma/B$, where ρ_r is in meters and B is in Tesla. For a typical ITER runaway electron, $\gamma\approx 20$, $B = 5$ T, and $a = 2$ m, so $|j_\perp/j_{||}| \approx 6.8\times10^{-3}$.

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