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Relativistic Particle Motion and Radiation Reaction in Electrodynamics

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Abstract: The problem of radiation reaction and the self force is the oldest unsolved mystery in physics. At times it is considered a minor issue, a malefactor born of classical electrodynamics, while at other times it is public enemy number one, a major inconsistency and unsolved problem. This work derives some of the basic and most important results while reviewing some of the other known approaches to the problem. Some historical notes are given, and yet another approach is discussed that accounts for radiation reaction without the unphysical behavior that plagues so many theories.

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1. Introduction

In recent years, with laser intensities between 10^{18} W cm^{-2} and 10^{22} W cm^{-2} created in the lab, many new, interesting, and potentially useful phenomena have been observed. In the so-called λ^3 regime, short pulses, generally in the near IR, are to be focused to a volume of a cubic wavelength. So far, the best that has been achieved in this "extreme light" quest is focusing to about three times that volume, but the true λ^3 pulse is on the horizon.[1] Thus, while it has long been expected that radiation reaction will become important on the extremely short time scale, now we see that the extreme intensity, which induces large time dilation, has made the inclusion of self forces imperative.

Using the wakefield phenomena, whereby electrons fall into synchronous acceleration

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with the pulse, electrons can be accelerated to velocities very near the speed of light.[1] This is due to the fact that the positive ions, being more massive, lag behind the electrons. This charge separation creates a huge electric field that propagates with, and accelerates, the electrons. This occurs on the centimeter scale, urging optical physicists and engineers to claim that kilometer sized linear accelerators may be replaced by table top laser systems!

Another common application of high intensity pulses occurs as the short burst of radiation is allowed to strike a surface, such as aluminum or tungsten, triggering a bounty of effects. One is high harmonic generation, creating 40 harmonics or more. Another is positron creation, resulting from the energetic electrons crashing back into the substrate, and near mono-energetic proton beams that are useful in medical treatments. All of these effects are discussed in a recent review article by some of the pioneers in this field.[1]

As these experiments approach pulse intensities of 10^{22} W cm^{-2} and higher, the problem of radiation reaction effects emerges like a lost relative expecting an inheritance. We are the nervous relatives gathered about, not sure what to do with him. In the past, he was sometimes considered a curiosity, the importance of whom obviated by his unreachability. But now he is at our doorstep, knocking, and we must admit this unwelcome visitor and let him enter into our equations and into our labs.

Early in the development of electrodynamics it was realized that accelerated charges radiate power, and Larmor derived the power radiated as,

$$P = m\tau_0 a^2 \tag{1}$$

where a is the acceleration of the particle of charge e, of mass m, and $\tau_0 = 2e^2/3mc^3$.[2] This result is valid as long as the velocity is small compared to the speed of light. Its correctness is proved daily by radio towers and related electromagnetic transmissions across the globe. However, problems begin when one considers that the electromagnetic field created by the accelerated particle can act on the charged particle that created this field. As will be discussed more fully below, this radiation field will act on the particle that created the field, in essence causing a self-interaction. Although radiation is an expected result, a self-interaction, in a way, is a surprising result. It is like the development of life on Earth, all the conditions are there, but its existence arises by itself, pulling itself up by the proverbial bootstraps. In other words, it is not assumed a priori that a self-interactions exists, it is not put into the action principle from which the equations are derived, it is a consequence of the theory. This result, that self-interactions are not included in the basic formulation of theory, has been called a formal inconsistency in the theory.[2] I will not get into that particular debate here, but I will follow the main approaches to this problem, i.e., dealing with the result and obtaining sensible equations of motion.

[2] I use cgs units throughout, except when I adopt dimensionless units for graphing purposes, for example.

1.1 Some major (and minor) steps

Lorentz was the first person to derive the self force. In 1909 the calculation shows up in Lorentz' book.[3] He writes, "By a somewhat laborious calculation it is found to be

$$(2e^2/3c^3)\ddot{V}.\text{ ''} \tag{2}$$

I have converted his result to cgs units, but the double dot was used by Lorentz and represents, as usual, the second time derivative. It is shown below that this term leads to unphysical runaway solutions, but Lorentz seemed to sideline these players. He went on to give the most commonly accepted view, saying, "In many cases the new force represented by (2) may be termed a *resistance* to the motion." To show this he considers the work done by this force and integrates by parts from t_1 to t_2, as is done below in (8), considering situations where the endpoint terms vanished (as in, for example, periodic motion). This derivation appears in most textbooks on this subject today. So, considering a velocity $V = b\cos nt$, Lorentz notes that the force is negative, a resistance force as he calls it, and uses $\ddot{V} = -n^2 V$ in (2).

Actually, in Lorentz' complete theory he models the electron by a small shell of charge, and derives a series for the self force. To be exact, the series would have to be infinite. Abraham was the first person to derive what we now call the LAD (Lorentz Abraham Dirac).[4] This is especially interesting since he did it before the advent of special relativity. (It is also referred to as the LD equation, but since Abraham derived it first, it seems to me that the A belongs, and lately LDE has gained some small popularity.)

With the subsequent development of quantum mechanics most physicists naturally apply quantum theory to radiation processes. But in 1938, before renormalization, they were still struggling with the infinite quantities ravaging perturbative quantum electrodynamics. Dirac felt that the problem might not be with quantum mechanics but with the model of the electron itself. He writes, "However, it seems more reasonable to assume the electron is too simple a thing for the question of the laws governing its structure to arise, and thus quantum mechanics should not be needed for the solution of the difficulty."[5] He eschews an apparently failed method of Born (who modified the Maxwell equations), and also argues that the Lorentz shell model of the electron must be abandoned, "... it seems that the Lorentz model has reached the limit of its usefulness and must be abandoned before we can make further progress."

In his approach, Dirac defines the actual field as the sum of the incident and retarded fields, and uses this to calculate the energy momentum tensor. He then expands the fields in a Taylor series and integrates this over a small tube near the worldline of the electron. The result he obtained is given below in (47) with a more heuristic derivation. Dirac points out that his result is the same as derived from the Lorentz theory, except that in the Lorentz case, there are an additional infinite number of higher order terms. In speaking of his result Dirac states, "But whereas these equations, as derived from the Lorentz theory, are only approximate, we now see that *there is good reason for believing them to be exact, within the limits of the classical theory.*"

While you might (feebly) argue that all Dirac did was to re-derive Abraham's result, Dirac did it for a point particle in a covariant way (whereas Abraham's model suffers from length contraction issues due to the electron's radius, rendering it unusable at relativistic speeds.) A more important issue is that Dirac used Maxwell's energy momentum tensor, a result derived for a continuous distribution of charge. Dirac applied it to a point electron where the field is singular along the worldline of the particle. Moreover, there is no *physical* justification for his rule of using the combination of (one half) the retarded and advanced field. There is however, a good *mathematical* reason–the singular parts cancel.

The other main problem with Dirac's result is that it suffers the same malady as the Lorentz equation, it leads to unphysical runaway solutions (discussed in detail section 2.). Dirac was unfazed by the pre-acceleration, stating, in the last sentence of his article "... the interior of the electron being a region of space through which signals can be transmitted faster than the speed of light." But another decade marked the triumph of renormalized QED and, as physicists were about to turn their attention to solid state matters and local gauge theories, the unsettled questions surrounding the equation of motion faded.

Faded but not vanished. Seven years later Wheeler and Feynman published what is now referred to as the absorber theory, in which the absorber plays an essential role in the emission process.[6] They attribute their approach to Tetrode,[7] who asserts that the sun would not shine if there were no one to see it ("The sun would not radiate if it were alone in space..."). Since the absorption of radiation occurs *after* emission, this theory uses both the retarded and advanced fields.[6] Wheeler writes, "The past was considered to be completely independent of the future. This idealization is no longer valid..."[8] Neither physicist dwelt on this extraordinary capitulation: Wheeler went on to make great strides in general relativity, bringing back to life a nearly forgotten field, and Feynman turned to quantum mechanics, making his well-known contributions including Feynman diagrams and the path integral formulation of quantum mechanics.

However, Wheeler and Feynman show that the combination of fields is not unique. In fact, as noted by Havas[9] and Rohrlich,[10] and discussed more fully by Teitelboim,[11] it is shown that the Dirac result may be obtained without using the advanced solution. By then, in their famous book,[12] Landau and Lifshitz derive an equation from the LAD equation using an iterative approach. This is called the LL equation and will be derived below.

In 1965 Rohrlich published his well-known book, *Classical Charged Particles.*[13] This gives many details and references on the derivations discussed above and includes more details on the history of self forces. The 1970s show a resurgence in the field of radiation reaction, in part, due to the excitement generated by pulsars at the time.[14] To kick off the decade, Mo and Papas[15] propose a brand new equation. They add a term that looks like the usual Lorentz force, except they replace the velocity with acceleration. For low velocity and a constant field this becomes tantamount to a redefinition of the mass. However, the authors provide no derivation, but instead rely on physical arguments that show their equation is sensible. Shen compares this equation to the LAD equation and states, “Therefore, we conclude, the new equation cannot lead to results physically distinguishable from the Lorentz-Dirac equation.”[16] On face value I would take this as a good thing, however, Shen actually uses the LL approach, so it is not really fair to state this is compared to the LAD equation. Shen also performs a detailed analysis of radiation for a particle zipping through a uniform magnetic field. He calculates radiation claiming to use the LAD equation, but again, when he solves the equations to lowest order, he essentially ends up using the LL approach.[17]

Still early in the 1970s, Steiger and Woods[18] investigate the motion of an electron in a high intensity field of circular polarization. To account for the radiation reaction, they ignore the longitudinal velocity and derive the power radiated in a cycle. From this they define a force and use this approach to calculate radiation effects. Since the longitudinal velocity becomes greater than the transverse velocity for high intensities (see Fig. 7 for linear polarization), this approach appears to be of limited value. They dwell on radiation reaction more in another publication.[19] In 1973 Herrera tackles the problem of the charged particle in a uniform magnetic field. He starts with the third order LAD equation, and uses an expansion in terms of the interaction parameter to obtain first order equations.[21]

In a faint echo of Dirac’s comment, Moniz and Sharp argue that it is essential to solve the runaway solution satisfactorily in order to establish the proper limit of QED.[22] They show that runaway solutions can be avoided with a non-zero radius electron, which, by now, comes as no surprise.

The next change in the landscape of radiation reaction occurred in 1991 by the publication of, what I refer to as, the Ford O’Connell (FO) equation, which I describe in section 4. [23] They derived the relativistic form of the equation of motion a couple of years later,[24] but also in 1991 argued that the electron should have structure,[25] which eliminates the violation of causality in the equation of motion. In this formulation, the Larmor formula is modified, and they use this result in a calculation of the total power.[26]

A new approach is introduced by Hartemann and Luhmann[27] in 1995. They calculate the radiation field of an accelerated charged particle and integrate the field over a sphere of radius R. They take a limit as $R \to 0$ and average over the spatial integration. This averaging process is not given any justification, but they obtain an interesting answer in which the radiation reaction force is in the (opposite) direction of the velocity.

Not long after this, Rohrlich’s voice is once again heard in the unequivocal title, “The correct equation of motion of a classical point charge.”[28]. He writes, “The LAD equation with its three serious defects has been used for about a century. It is time to replace it by a correct equation.” He referred to a treatment by Spohn[29] that I mentioned, and ends up with the LL equation, which he claimed was exact: “The simple physical argument above thus has resulted in the correct and exact equation of motion...,” at which point he refers to the LL equation.

However, in another bold statement Bosanac claims “The relativistic dynamics of a charge in the presence of the radiation reaction force is solved in general.”[30] This work is predicated upon the assumption that mass energy of the charge can be converted to energy of the electromagnetic field, an assumption weakened by its lack of independent confirmation. Others[31] attempt to break the radiation reaction force into a part that resides in the stress of the field created by the charge.

In the history of physics we have sometimes found that difficult and abiding problems are actually the result of too narrow a focus of view. For example, the theory electromagnetism must contain both electricity and magnetism to be relativistically correct. Going further, the weak force is correctly understood (as best as today allows) through a unified, or combined, theory of electromagnetism and the weak force.

In this vein, there has been the consideration of magnetic dipole radiation as well as the radiation from the electric charge.[32] It is shown that, for a changing magnetic moment, radiation fields are fourth order in the expansion of the retarded potential. Beyond this, several authors have looked at more general formulations of electromagnetism. This is where the concept of the magnetic monopole enters our little tour of radiation reaction. Generalizing the field equations of electrodynamics to include magnetic charge, the authors show several advantages to the symmetrized electrodynamics.[33] In the end, however, these authors re-derive the Dirac result, except there is a new coupling constant for the magnetic charge. The effects of magnetic charge on the radiation reaction were also investigated by Heras.[34] In another generalization, the problem has been considered in various dimensions, with special emphasis comparing an even number of dimensions to an odd,[35] and to renormalization techniques.[36] In more modest cases, the effect of radiation reaction is considered for the case of classical elliptical orbits in hydrogen,[37] and also for a plasma.[38] In this, the authors adapt the method of Steiger and Woods described above.

At about this time, Blinder[39] revives the notion of the finite size electron and, as is usual in these models, avoids the runaway solutions that otherwise hound us like the Furies. In the final result, the force is evaluated at a retarded time. Another interesting result appears at this time in which the LAD equation is re-derived, but unlike Dirac's integration of the stress tensor along the world tube of the particle, the authors revert to a finite distribution of charge of size ϵ.[40] They use a form of mass renormalization which enables them to obtain a finite limit as $\epsilon \to 0$.

More recently Medina[41] adopts the finite size electron and carefully examines all of the forces, including the stresses. He claims, "The problem of the self-interaction of a quasi-rigid classical particle with an arbitrary spherically symmetric charge distribution is completely solved up to the first order in the acceleration." In this model, the electron cannot be smaller than a size of the order of the classical electron radius. Although we know the electron is, in fact, much smaller than this, for classical physics applications this is probably not a concern. For example, the wavelength of visible light is orders of magnitude larger than this. Medina shows that the FO type of equation Rohrlich derives is a limiting case.[42]. In a very recent article, Rohrlich states, "Using physical arguments, I derive the physically correct equations of motion for a classical charged particle from the Lorentz-Abraham-Dirac equations..."[43] His method is based on the premise stated in the following, "Consequently, in order to be able to use differential equations of motion the external force must vary slowly enough over the size of the charge distribution so that it will not be able to distinguish between a small but finite particle radius and a point particle." With this, he ends up with a FO type equation. I will discuss this more fully below.

Last year the problem was analyzed by Gralla, Harte, and Wald.[44] They consider a model in which the charge and mass of a particle scales in specified way such that the mass and charge go to zero as the size of the particle goes to zero. Their results include magnetic and electric dipole results as well as radiation reaction. They reduce the result to the LAD equation in their perturbative limit.

1.2 Uniform acceleration

In the year Einstein finally published his theory of gravity, general relativity was born. It was 1915 and special relativity was a mere 10 years old. It was finally being taken seriously, although quantum mechanics was a future child not yet conceived. It was at this time that G. A. Schott writes, "During a theoretical investigation of the origin of X-rays I found it necessary to take into account the effect of the motion of the electron of the reaction due to its own radiation..."[45] He goes on to develop the equations of hyperbolic motion, which refers to the constant force problem (constant acceleration in the particle's rest frame). The older history is documented in the article by Fulton and Rohlich[46] quoted above, who give Born credit for the first calculation of hyperbolic motion. Since then there has been an often contentious controversy over this issue. I will not document this long and lively history, but mention a few papers from which the reader can delve further into the literature.

Referring to the problem of radiation emitted by a charge suffering a constant acceleration, Cohn[47] sums up the situation in 1976, "Solutions to the problem range from the early conclusion of Pauli that no radiation is emitted to more recent statements, such as that of Rohrlich's, which claim that the radiation rate is constant as indicated by the well-known Larmor relation." Cohn goes on to demonstrate why Pauli got the wrong answer, arguing that "the radiant energy flux is not given by the flux of the total Poynting vector..." Cohn builds on the work on Drukey[48] who shows the fallacy that the radiation is zero in *every* reference frame by comparing the fallacious reasoning to Zeno's argument that nothing can move.

More recently Sorkin[49] gives a nice perspective on the issue. At the risk of quoting the entire opening paragraph, he states, "A well-known peculiarity of the radiation reaction force on a charged particle is that it vanishes when the particle accelerates uniformly. But this raises a paradox. An accelerating charge radiates, and the longer the acceleration continues, the greater the total energy radiated. If one asks where this energy comes from in the case of uniform acceleration, the usual answer is that it is "borrowed" from the near field of the particle and then "paid back" when the acceleration finally ceases. But this "debt" can be arbitrarily great if the acceleration remains uniform for a long enough time. What, then, if the agent causing the acceleration decides not to repay the borrowed energy? What if, in fact, it does not even possess enough energy to pay its immense debt at that time? If we believe in conservation of energy, the respective answers must be that the accelerating agent must not be at liberty to avoid transferring the required energy and that it must always possess the necessary amount to cover its accumulated debt." This terminology is unnervingly topical, but makes the point.

In the same year, a derivation of the LAD equation appears based on QED.[50] Their result is based upon a particular photon emission rate they derive by using a "WKB wave function" to leading order in $\hbar$. In the following year the LAD equation is declared "...one of the most controversial equations in the history of physics."[51] These authors, as mentioned above, show that the LL, like the LAD equation, has the unphysical property of radiating energy while not affecting the motion of the particle.

2. Non-relativistic theory

To preview the discussion of radiation reaction I will call on an analogy. Imagine pushing a puck along a frictionless table. Assuming that the normal force balances gravitation, we have, in the plane of the table, $\boldsymbol{F} = m\boldsymbol{a}$, where m is the mass of the puck, and $\boldsymbol{F}$ is the force applied to the puck. Now turn the friction on and the acceleration will obviously be different. The equation of motion becomes

$$m\boldsymbol{a} = \boldsymbol{F} + \boldsymbol{f} \tag{3}$$

where $\boldsymbol{f}$ is the *added force*, the force that will account for the friction. Restricting to one dimensional motion, we define f as the force that acts on the puck, the friction force. To find f, we start with the fact that the work done by the puck is proportional to its weight times the distance it moved, μgmL, where L is the distance it moved. We know that the work done by the puck (which is negative) equals the work done by friction. However, the work done by f, the force *on* the puck, is the negative of the work done *by* the puck. (The work done on the puck is the force on the puck times the distance it moves.) Thus, the negative of the work done on the puck is equal to the work done by friction. Writing this out, and using $V = dx/dt$, we have

$$-\int fV dt = \mu mgL \tag{4}$$

from which we find, putting f outside the integral,

$$f = -\mu mg, \tag{5}$$

which we knew all along. The reason this works is because the work done on the puck, and therefore the associated power, is known.

Now we apply this to electromagnetism, where again we know the power radiated, which is given by (1). So again, we add a force to account for radiation reaction (which replaces friction)

$$m\boldsymbol{a} = \boldsymbol{F}_e + \boldsymbol{f}_r \tag{6}$$

and, as before, assume that the work done *by* the electron is equal to the energy radiated, or equivalently, the negative of the work done *on* the electron is equal to the energy radiated. Writing this in one dimension we have,

$$-\int f_r dx = \int P dt \tag{7}$$

or, integrating by parts,

$$-\int f_r V dt = m\tau_0 \int \dot{V} dV = m\tau_0 \left(\dot{V}V)|_{t_1}^{t_2} - \int V\ddot{V}dt \right) \tag{8}$$

where $\dot{V} = dV/dt$.

Usually we consider the case that either the acceleration or the velocity vanishes at the endpoints t_1 and t_2, so that the integrated term vanishes. With this we see that the radiation reaction force is given by,

$$f_r = m\tau_o \ddot{V}, \tag{9}$$

and that therefore the equation of motion is, reverting to three dimensions,

$$m\boldsymbol{a} = \boldsymbol{F}_e + m\tau_0 \ddot{\boldsymbol{V}}. \tag{10}$$

This is the non-relativistic equation of motion with the radiation reaction effects. It based on the simple notion of conservation of energy, that the work done by the radiation field (which is the negative of the work done by the particle) is simply the time integral of the power radiated. It is a very basic and intuitive idea, and at first one may not think to question such an obvious procedure. But the result given by (10) has such weird consequences, that for many decades physicists have sought a better solution. For decades we have been living with an equation that cannot be wrong, yet cannot be right.

As a well-known example of the wild behavior, let us consider an electron with initial velocity V_0 with no external forces. The equation of motion, with radiation reaction is,

$$\dot{V} = \tau_0 \ddot{V} \tag{11}$$

and the solution is $V = V_0 e^{t/\tau_0}$. For an electron, τ_0 is of the order of 10^{-23} s, so in a very short time the particle flies off approaching the speed of light, clearly unphysical. Clearly unacceptable.

Strictly speaking, I have cheated. This kind of solution violates the conditions that I assumed to hold in the derivation, that the integrated terms vanish at the endpoints. Since this motion is neither periodic nor had zero acceleration or velocity at t_2, the result (11) is not valid. However, when this is re-derived relativistically (below), we will see the issue does not go away.

A well-known technique used to solve (10) is to use an integrating factor by defining $\dot{V} = ue^{t/\tau_0}$ which gives[53]

$$\dot{V} = -\frac{e^{t/\tau_0}}{m\tau_0} \int_{t_0}^{t} F(t')e^{-t'/\tau_0}dt' \tag{12}$$

where t_0 is a constant. To find it, we may look at the special case that F is a constant, then (12) gives

$$m\dot{V} = F(1 - e^{\frac{t-t_0}{\tau_0}}) \tag{13}$$

which shows that, to agree with Newtonian physics, $t_0 = \infty$. Using this, and letting $s \equiv (t' - t)/\tau_0$, we may put this in the final form,

$$\dot{V} = \frac{1}{m} \int_{0}^{\infty} F(t + s\tau_0)s^{-s}ds. \tag{14}$$

To help us understand (14), we may expand F in a Taylor series, since τ_0 is small, $F(t + s\tau_0) = F(t) + s\tau_0\dot{F}(t)$, neglecting higher order terms. The integrals are elementary and reduce to unity, so (14) becomes,

$$m\dot{V} = F(t) + \tau_0\dot{F}(t). \tag{15}$$

Another way of writing this may be obtained by noting that the right hand side is the Taylor series of $F(t + \tau_0)$, neglecting higher order terms, so that we obtain the shocking equation,

$$m\dot{V} = F(t + \tau_0), \tag{16}$$

or equivalently, with $\dot{V} = a$,

$$ma(t - \tau_0) = F(t). \tag{17}$$

This shows that the electron must be prescient. It accelerates *before* the force is applied. For example, if a unit step pulse were applied at $t = 0$, the electron would begin its acceleration at a time τ_0 *before the pulse arrived.* As Wheeler and Feynman wrote, "Pre-acceleration and the force of radiation reaction which calls it forth are both departures from that point of view of nature for which one once hoped, in which the movement of a particle at a given instant would be completely determined by the motions of all other particles at earlier moments."[6] One other attempt, however, was tried in 1976 which concerns specific cases, such as two identical particles approaching each other.[54] This procedure both avoids the runaway miscreants while at the same time allows for radiation. In this method, the authors are not shackled by the conventional sign of time. In their words, "A way around the problem of runaway solutions is to integrate the third-order equation *backward* in time. The runaway contribution is then rapidly damped to zero..." This is no less unphysical than pre-acceleration so, even though runaways are eliminated, the cost is high.

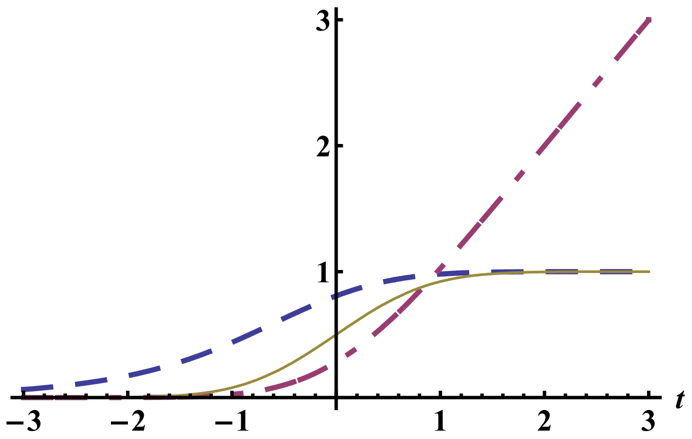

Fig. 1 The solid line is the force, the dashed line is $V(t)$, and dot-dashed is the solution $V_0(t)$ (without radiation reaction), for $\tau_0 = 1 = m$.

For these reasons, in this war between radiation reaction theory and physically acceptable solutions, we are faced either with runaway solutions or solutions that violate causality. Since this violation is tiny, we have learned to make an uneasy truce in the battle, and use (14) for many non-relativistic situations.

However, there is an interesting counter argument. A force represented by a step function is just as unphysical as "pre-acceleration." So let us consider a more realistic pulse,

$$F(t) = mg\frac{1+\mathrm{erf}(\mu t)}{2}, \tag{18}$$

which becomes the unit step (times mg) in the limit that $\mu \to \infty$. Let us consider the solution to the problem without radiation reaction, i.e.,

$$m\dot{V}_0 = F(t) \tag{19}$$

and compare this to $V(t)$, the solution of (14) with (18) with $m = 1$, $g = 1$ and $\tau_0 = 1$ for ease of viewing. These equations may be integrated in terms of error functions, but a graph of the solutions is more illustrative, which is given by Fig.1. In Fig.(1), we see that V, the solution to (14), is well behaved (V levels off due to the extremely large value I chose for τ_0). One might try to argue that there is causality violation, but this is not the case. This graph shows that V is simply larger than V_0, or that the velocity is larger, initially, than what the velocity would have been without radiation reaction. This is surprising, but violates no principles we cherish. Of course, a Gaussian pulse is not physical because it extends to minus infinity. However, a step force (or any force for which the derivative is discontinuous) is equally unphysical.

This result, however, does not save the day. It is not so simple for the relativistic case treated below, although there has been an attempt,[55] and a formal solution has been given by Hartemann.[56]

3. Relativistic equation

All of the above results are born with the original sin of Newtonian physics and are therefore suspect. In this section the equations are generalized to the relativistic case, and we adopt the notations and conventions of Jackson.[53] Spacetime is represented by the Minkowski metric tensor

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{20}$$

The electromagnetic field tensor is defined, using the comma notion

$$A_{\mu,\nu} \equiv \frac{\partial A_\mu}{\partial x^\nu} \tag{21}$$

by $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ where $A_{\nu,\mu} \equiv \partial A_\nu / \partial x^\mu$ and where A_μ is the four potential. The field tensor is given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{22}$$

The four velocity is $v^\sigma = dx^\sigma/d\tau$ where τ is the proper time, and the traditional equation of motion of a charged particle in an electromagnetic field, without radiation reaction, is given by

$$\frac{dp^\mu}{d\tau} = \frac{e}{c} F^{\mu\sigma} v_\sigma \tag{23}$$

where $p^\mu = mv^\mu$.

Before we make our final assault on the equation of motion with radiation, we will consider the relativistic equation of motion without radiation reaction, (23), and consider a plane electromagnetic pulse polarized in the x direction, which is given by

$$\boldsymbol{E} = Eh(kz - \omega t)\hat{\boldsymbol{x}} \tag{24}$$

where E is constant and h describes spatial and temporal dependence. Since it is a function of $z - t/c$, it satisfies Maxwell equations (with $c = \omega/k$). The magnetic field is

$$\boldsymbol{B} = Eh(kz - \omega t)\hat{\boldsymbol{y}}. \tag{25}$$

It is sometimes helpful to write the equations in non-dimensional form. We let $kx^\mu \to x^\mu$ and $\omega t \to t$, and $F^{\mu\nu} = Ef^{\mu\nu}$. For example, $\cos(kz - \omega t) \to \cos(z - t)$. In non-dimensional form, the equation of motion becomes

$$\frac{dv^\mu}{d\tau} = af^{\mu\sigma}v_\sigma \tag{26}$$

where the dimensionless constant $a = qEL/mc^2$ and $L = 1/k = \lambda/2\pi$. This constant, a, is a measure of the onset on relativity. We can think of the numerator qEL as the work done on the electron during a time period of the order of a cycle. If this is small compared to the rest mass, the denominator, then $a << 1$ and the relativistic effects are small. The transition point occurs for an intensity just over 10^{17} W cm^{-2}, the value for which $a = 1$. This letter, a, is commonly used in the literature, although in some of the older literature it was referred to as γ_0. In terms of the potential A^σ, for a plane wave we have $A = E/L$ (for the relevant component) and we can express a in another common form, $a = eA/mc^2$, which also clearly shows the physical significance of this parameter: It is the potential energy divided by the rest energy. Also in the literature, a common litmus for relativity is the "quiver energy," which is defined as the average value of the non-relativistic kinetic energy. Solving $F = ma$ using the force $eE\cos\omega t$, we have $< KE >= e^2E^2/4m\omega^2$. Setting the quiver energy to the rest energy of the electron gives $mc^2 = eEL/2$, within a factor of two as what we had above.

In order to solve (26) we write out the four equations:

$$\frac{dv^0}{d\tau} = ahv^1 \tag{27}$$

$$\frac{dv^1}{d\tau} = ah(v^0 - v^3) \tag{28}$$

$$\frac{dv^2}{d\tau} = 0 \tag{29}$$

$$\frac{dv^3}{d\tau} = ahv^1. \tag{30}$$

We see that (29) shows that v^2 is constant, so we will take it to be zero. We also see that (27) and (30) have the same right side, so we obtain

$$v^0 = 1 + v^3. \tag{31}$$

This is an extremely important result, for by integrating it we obtain $\tau = t - z$. Since the forces are functions of $z - t$, this may be replaced by $-\tau$ which allows for direct integration. Also, with this (28) can be integrated to give

$$v^1 = a\int h(-\tau)d\tau. \tag{32}$$

Finally, (27) and (32), with (31) imply that $\dot{v}^0 = v^1\dot{v}^1$ which may be integrated to give

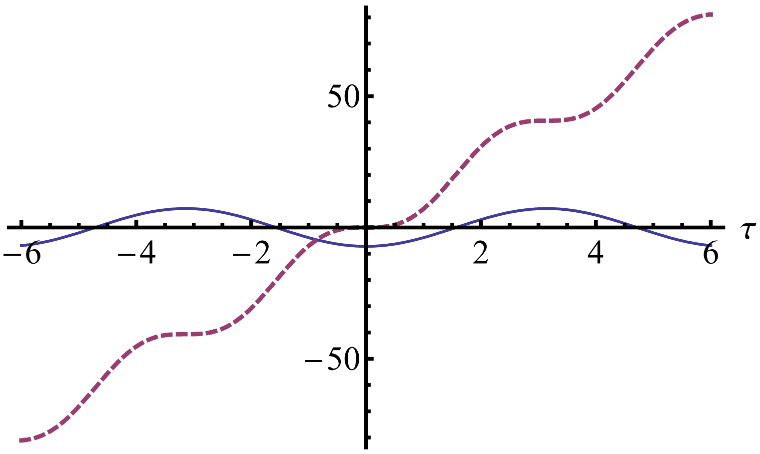

Fig. 2 Displacement in the x and z direction (dashed)

$$v^0 = 1 + (v^1)^2/2. \tag{33}$$

Thus we have a complete solution, given by (31), (32), and (33) in terms of the quadrature of h, a known result.[57]

The simplest case is instructive, if unrealistic (for high intensity): In this, the electromagnetic field is purely sinusoidal so that $h(z-t) = \cos(z-t)$. The integrations are elementary and we find

$$x = -a^2 \cos(\tau) \tag{34}$$

and

$$z = \frac{a^4 \left(\frac{\tau}{2} - \frac{\sin(2\tau)}{4}\right)}{2}. \tag{35}$$

It is seen that there is a drift velocity in the z direction, which gives rise to the secular term in (35) (proportional to τ). These are illustrated in Fig. (2). It is interesting to note from (35), and the graphs, that there is a natural nonlinearity in the equations of motion due to relativistic effects. The frequency of oscillation in the longitudinal (z) direction is twice that of the transverse direction.

It is sometimes useful to make a parametric plot of x and z in terms of τ, which is shown in Fig. (3). It is interesting to plot the orbit in a reference frame that is moving at the drift velocity of the electron. This gives the well-known figure-8 pattern (or the infinity symbol), which is shown in Fig. (4).[58]

It will also be instructive to consider a short pulse of electromagnetic radiation, given by

$$h = \frac{e^{-((z-t)/w)^2}}{w} \cos(\Omega(z-t)) \tag{36}$$

where the dimensionless Ω determines the frequency and the dimensionless w determines

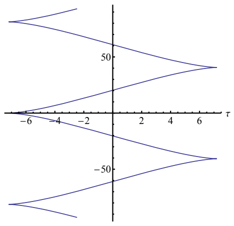

Fig. 3 Parametric plot of x and z vs. time

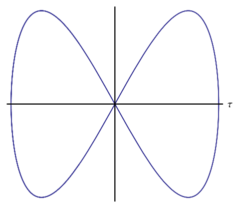

Fig. 4 Parametric plot of x and z vs. proper time in a reference frame moving at the drift velocity

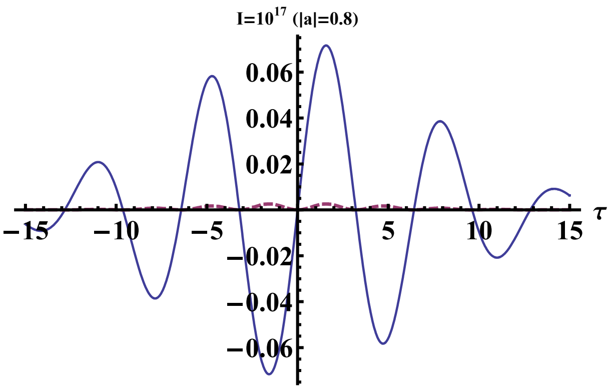

Fig. 5 The transverse (solid) and longitudinal (dashed) components of the four velocity at $I = 10^{17}$ W cm^{-2}.

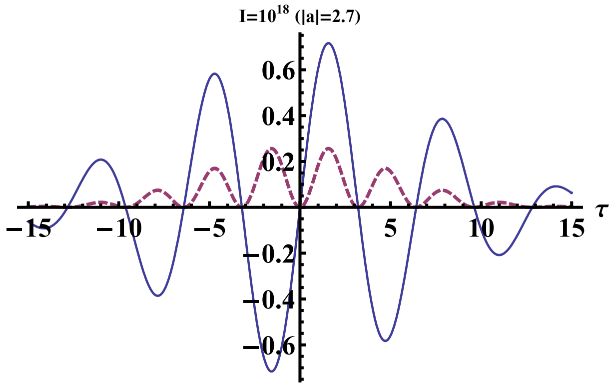

Fig. 6 Components of the four velocity at $I = 10^{18}$ W cm^{-2}.

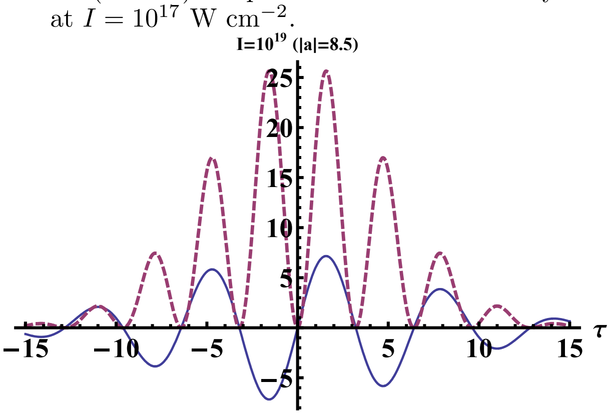

Fig. 7 Components of the four velocity at $I = 10^{19}$ W cm^{-2}

the width of the Gaussian.[3] The result is plotted in (5)-(7) for $\Omega = 1$ and $w = 10$, which correspond to a pulse consisting of a few cycles of visible light.

The solid line is the transverse component, v^1. The longitudinal component, v^3, is in the direction of propagation and is due to the effect of the magnetic field. The magnetic field makes a small contribution below $I = 10^{17}$ W cm^{-2} and signals the onset of relativistic effects. Equivalently, $a << 1$ signals the non-relativistic regime while

[3] Actually, it is not necessary to introduce both Ω and w, since we already have room to scale things in terms of L, but it makes things more transparent.

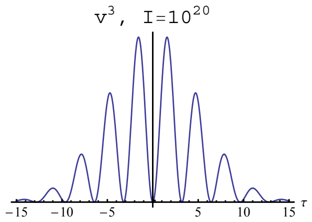

Fig. 8 The longitudinal component of the four velocity at $I = 10^{20}$ W cm^{-2}

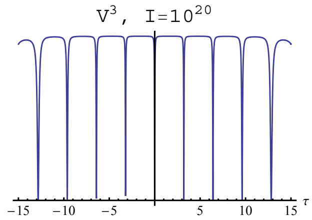

Fig. 9 The corresponding ordinary velocity.

$a >> 1$ is relativistic.

Most of the time the results derived here are left in terms of proper time, although in practice one often needs the results in the lab time. This transformation can be performed by integrating (27), which gives t as a function of τ. Also, often one is interested in the ordinary velocity instead of the four velocity. These are compared in Figs. 8 and 9 for the z component of the velocity.

The integral (32) with (36) may be found directly by completing the square and is given by the function $\mathcal{E}(\tau)$,

$$v^1 = a\mathcal{E}, \tag{37}$$

where $\mathcal{E}(\tau)$ is given in terms of the error function:

$$\mathcal{E}(\tau) = \frac{\sqrt{\pi}}{4} e^{-\Lambda^2} \left(2 + \mathrm{erf}(\tau/w + i\Lambda) + \mathrm{erf}(\tau/w - i\Lambda)\right) \tag{38}$$

where $\Lambda = \Omega w/2$. This assumes the initial condition that $v^1(-\infty) = 0$. Since $\mathcal{E}$ is of form $z + z^*$, we know that it is real.

This result may be transformed to an interesting and useful series by continued integrating by parts. With $\zeta = \tau/w$ and $d_n \equiv \frac{d^n}{d\zeta^n} e^{-\zeta^2}$,

$$\mathcal{E}(\tau) = \sin\Omega\tau \sum_{n,\text{even}} (-1)^{n/2} \frac{d_n}{(2\Lambda)^{n+1}} + \cos\Omega\tau \sum_{n,\text{odd}} (-1)^{(n-1)/2} \frac{d_n}{(2\Lambda)^{n+1}}. \tag{39}$$

This is useful for $\Lambda > 1$. This shows that the (x component of the) velocity is sinusoidal with an exponential envelope function, a result that is not self evident from (38). This series is very robust, and gives an excellent approximation keeping only the first term in the sum when Λ is bigger than unity. For example, Fig. 10 depicts a graph comparing v^1 using (38) and (39) using only one term in the sum with $\Lambda = 5/2$. For $\Lambda = 5$ the graphs are barely discernible.

Before we go on to the radiation reaction effects, there is a theorem that has come to be of considerable interest in the particle acceleration community called the Lawson-Woodward theorem.[59] In general terms it states that a charged particle cannot gain energy from an electromagnetic wave. A plane wave can break this rule, depending on the envelope. For example, the net x component of the velocity for the example above is

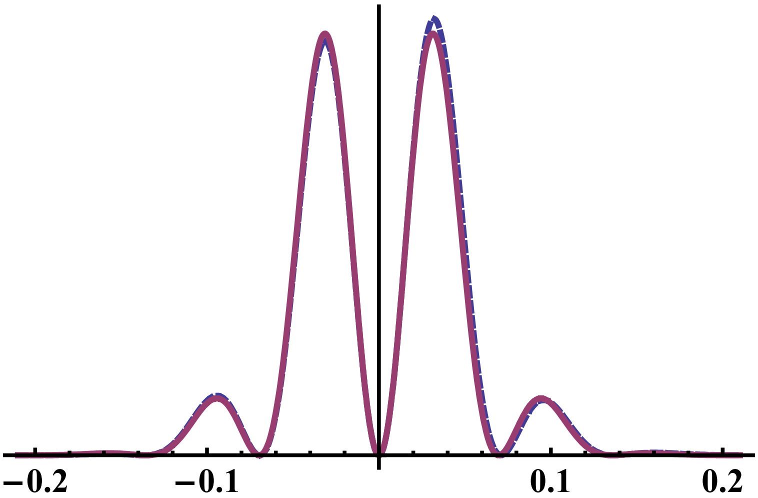

Fig. 10 v_1 using (38) (solid), and (39) (dashed), for $I = 10^{20}$ W cm^{-2}. Proper time is plotted on the horizontal axis.

$$\Delta v^1/c = a \int_{-\infty}^{\infty} h(-\tau)d\tau = \sqrt{\pi} a e^{-\Lambda^2}. \tag{40}$$

Alternatively, the four potential is given by $A^\mu = (0, \phi, 0, 0)$ $(A_1 \equiv \phi)$ where

$$\phi = -EL\mathcal{E}. \tag{41}$$

The kinetic energy of the particle is

$$\mathrm{K} = (\gamma - 1)mc^2 = (v^0 - 1)mc^2 = v^3 m = \frac{m}{2}(v^1)^2 \tag{42}$$

where (31)-(33) were used. Using (37) and (41) one may also show that this may be put in the form, defining the $U = e\phi$,

$$\mathrm{K} = \frac{1}{2}(U(\infty) - U(-\infty))^2. \tag{43}$$

In the modern version of the Lawson-Woodward theorem the proviso $U(\pm\infty) \to 0$ is required, so that there really is no violation of the theorem.

However, Fradkin showed that when the radiation reaction is taken into account, the particle does indeed gain energy, a result to be confirmed below.[60] For a flavor of some of the controversy one may read the literature,[61] and one may also consult [62].

4. Radiation reaction

In the following I will use heuristic derivations that focus on conservation of energy. For methods that derive the electromagnetic field of the electron, one may consult the literature given above, especially Dirac[5] and Rohrlich[13].

4.1 Basic derivations

In order to find the relativistic equation of motion with radiation reaction, we use the generalization rule that the ordinary velocity is replaced by the four velocity, thus $\boldsymbol{V} \to v^\sigma$ and $\boldsymbol{V} \cdot \boldsymbol{V} \to -v_\sigma v^\sigma$. The minus signs shows up because for $n = 1, 2, 3$ $v_n = -v^n$.[4] With this the power scalar from (1) becomes

$$P = -m\tau_0 \dot{v}_\sigma \dot{v}^\sigma \tag{44}$$

where, from now on, the dot denotes the derivative with respect to proper time. Now we generalize (23) by adding the yet unknown radiation reaction force f^μ,

$$m\frac{dv^\mu}{d\tau} = \frac{e}{c}F^{\mu\sigma}v_\sigma + f^\mu \tag{45}$$

and set about finding f^μ. If we generalize the force used in (10), $m\tau_0\ddot{\boldsymbol{V}} \to m\tau_0\ddot{v}^\mu$, we obtain,

$$\frac{dv^\mu}{d\tau} = \frac{e}{mc}F^{\mu\sigma}v_\sigma + \tau_0\ddot{v}^\mu. \tag{46}$$

Unfortunately this cannot be correct.

This is because the four velocity satisfies the constraint $v_\sigma v^\sigma = c^2$. By differentiating this we find $\dot{v}^\sigma v_\sigma = 0$. This result states that the four acceleration is orthogonal to the four velocity, a result that will be used below. Differentiating this result, we find that $\dot{v}^\sigma \dot{v}_\sigma = -v_\sigma \ddot{v}^\sigma$, another result to be used below. Now, we multiply (46) by v_μ (and sum over μ) and note that the left side vanishes, and that $F^{\mu\nu}v_\mu v_\nu$ also vanishes because $F^{\mu\nu}$ is antisymmetric while $v_\mu v_\nu$ is symmetric in μ and ν. But $v_\sigma \ddot{v}^\sigma$ does not vanish, so the equation is inconsistent.

At this point we can throw our hands up in defeat, or try to figure out how to salvage (46). For now I will show how to repair this equation, but later I will argue that perhaps we should have thrown our hands up after all. It is easy to fix: We simply add another term to make the equation consistent. This is valid as long as it disappears in the low velocity limit. It is easy to check that the added term should be $\tau_0 v^\mu \dot{v}_\sigma \dot{v}^\sigma / c^2$ so the equation becomes

$$\frac{dv^\mu}{d\tau} = \frac{e}{mc}F^{\mu\sigma}v_\sigma + \tau_0\left(\ddot{v}^\mu + \frac{v^\mu}{c^2}\dot{v}_\sigma\dot{v}^\sigma\right). \tag{47}$$

This is the celebrated Lorentz Abraham Dirac equation, or LAD, which was also discussed in the introduction.[5]

A formal procedure to derive this may be viewed as follows. We generalize the work done, $\int \boldsymbol{F} \cdot \boldsymbol{V} dt$ to $-\int f^\sigma v_\sigma d\tau$ (the trouble here is the breakup of the space part and

[4] I use $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$.

[5] This equation appears in the literature with different signs in the radiation reaction term, even changing from edition to edition of the same book. I use the convention that $ds^2 = c^2d\tau^2 - dx^2 - dy^2 - dz^2$, while some authors use the negative of this.

the time part, the latter carrying information about power, but we assume that in the nonrelativistic limit the interpretation is correct), then, as before, we assume that "the negative of the work done on the electron is equal to the energy radiated," so that

$$\int v_\sigma f^\sigma dt = -\int P dt. \tag{48}$$

Integrating by parts this may be written

$$\int d\tau v_\sigma \left(f^\sigma - m\tau_0 \ddot{v}^\sigma\right) = -m\tau_0 v_\sigma \dot{v}^\sigma|_{\tau_1}^{\tau_2}. \tag{49}$$

Since $v_\sigma \dot{v}^\sigma$ vanishes we can read out f^σ from (49), so that (45) becomes (46).

If we write this as

$$m\frac{dv^\mu}{d\tau} = \frac{e}{c}F^{\mu\sigma}v_\sigma + \mathcal{G}^\mu, \tag{50}$$

then $\mathcal{G}^\mu$, given by

$$\mathcal{G}^\mu = +m\tau_0\left(\ddot{v}^\mu + \frac{v^\mu}{c^2}\dot{v}_\sigma\dot{v}^\sigma\right), \tag{51}$$

is called the self force, which is sometimes called the von Laue four vector and sometimes called the Abraham vector. We may break it up as

$$\mathcal{G}^\mu = F^\mu_{\mathrm{S}} + F^\mu_{\mathrm{R}} \tag{52}$$

where the Schott force is

$$F^\mu_{\mathrm{S}} = m\tau_0\ddot{v}^\mu \tag{53}$$

and the radiation reaction force

$$F^\mu_{\mathrm{R}} = m\tau_0\frac{v^\mu}{c^2}\dot{v}_\sigma\dot{v}^\sigma. \tag{54}$$

This nomenclature is given in Ref. [63], page 1110, with references to the early work of Abraham, Schott, and von Laue, but one may also see [64]. This states that the *self force* is equal to the *Schott force* plus the *radiation reaction force*. However, not all authors make this distinction, and while most correctly refer to the Schott force, many authors let "radiation reaction force" refer to the entire self force, not to mention those who use the term radiation damping. As with the sign convention, reader beware! A very nice and relatively recent account of this equation, along with a full discussion of the Schott term, is given by Eriksen and Grøn.[65]

The Schott term is a total derivative, and thus represents a reversible process. The radiation force is obviously not of this form and represents an irreversible loss of energy and momentum due to the radiation. As (54) shows, this force is in the opposite direction of the velocity (considering the spatial part). This does not mean that the Schott term has nothing to do with energy loss, however. For example, if it is ignored one may find paradoxical energy losses, as noted by [66], and resolved by [67].

Defining $\Omega = eE/mc$ which is used below, the LAD equation is,

$$\frac{dv^\mu}{d\tau} = \Omega f^{\mu\sigma} v_\sigma + \tau_0 \left(\ddot{v}^\mu + \frac{v^\mu}{c^2} \dot{v}_\sigma \dot{v}^\sigma \right). \tag{55}$$

Unfortunately it is unphysical, and therefore must be wrong. Due to the third derivative term ($\ddot{v}^\mu$) it suffers the same runaway solution as the non-relativistic version. It may be hoped that the extra relativistic term tempers the runaway solution, but it does not. Even worse, the non-relativistic result (12) obtained by using an integrating factor does not always work in the relativistic case, so we are even worse off (an attempt, though, at generalizing (12) may be found in the literature),[55] and one should also consult Hartemann's book.[56] Moreover, it has recently been shown that, except in the weak field case, the LAD does not agree with the theory of Compton scattering.[68] However, many authors use an iterative approach that sidesteps the third derivative term, which is described in detail below.

By the way, the $\ddot{v}^\mu$ term also signals another problem. If the force is discontinuous, then $\ddot{v}^\mu$ is singular. Real forces are not discontinuous, but many examples use this artifice. An example is the release of a particle from rest in a uniform field, where we would normally assume the net force is zero until the time of release, at which time it instantaneously acquires some non-zero value. This pesky issue will haunt us a little later.

When physicists are stuck in this sort of predicament, we are forced to carefully examine our most fundamental assumptions, and one I glossed over starts with (23). Although it is true that $v_\sigma \dot{v}^\sigma = 0$, it is only an assumption that

$$v_\sigma \frac{dp^\sigma}{d\tau} = 0 \tag{56}$$

(i.e., it is assumed that the mass is constant). So, starting from (23), and assuming there is an additional force due to radiation reaction, we may equate that to the power, as before, to obtain,

$$\frac{dp^\mu}{d\tau} = \frac{e}{c} F^{\mu\sigma} v_\sigma + m\tau_0 \ddot{v}^\mu \tag{57}$$

Now we multiply (and sum) by v_μ to find $\dot{m}c^2/m = -\tau_0 \dot{v}^\sigma \dot{v}_\sigma$ and

$$\frac{dv^\mu}{d\tau} = \frac{e}{mc} F^{\mu\sigma} v_\sigma + \tau_0 \left(\ddot{v}^\mu + \frac{v^\mu}{c^2} \dot{v}_\sigma \dot{v}^\sigma \right) \tag{58}$$

which looks like a clone of the LAD equation, but its DNA is much different, since the (rest) mass is not constant.[6] I touched on this in section 1.1, but since the mass of electrons always is measured to be the same, this idea cannot hold for elementary particles.

[6] In older literature this is called the rest mass, and in that parlance, there is a "relativistic rest mass increase."

Exact solutions to (55) have not been found, except for the simple case of a constant field (this perplexing issue is discussed below). Therefore we seek some sort of perturbative solution. A common method of attack is to look for asymptotic solutions of the form

$$v^\sigma = {}_0v^\sigma + \tau_0({}_1v^\sigma) + \mathcal{O}(\tau_0)^2, \tag{59}$$

assuming the second term is small compared to the first. If (59) is used in (47) and each order of τ_0 is equated, two equations arise. The $\mathcal{O}$ (τ_0^0) equation is (23) and the other is the $\mathcal{O}(\tau_0^1)$ equation. These equations may be combined as

$$\frac{dv^\mu}{d\tau} = (e/mc)F^{\mu\sigma}v_\sigma + \tau_0\left((e/mc)\dot{F}^{\mu\sigma}v_\sigma + (e/mc)^2(F^{\mu\gamma}F_\gamma{}^\phi v_\phi + F^{\nu\gamma}v_\gamma F_\nu{}^\phi v_\phi v^\mu)\right) \tag{60}$$

which is valid to $\mathcal{O}(\tau_0^1)$.

This is another celebrated equation, the Landau Lifshitz, LL, equation,[12] which had been used extensively over the years when radiation reaction terms are included. An example of recent usage may be found in the literature.[69] This procedure may also be used in the non-relativistic case where

$$m\dot{\boldsymbol{V}} = \boldsymbol{F} + m\tau_0\ddot{\boldsymbol{V}}. \tag{61}$$

Since the radiation reaction force $\tau_0\ddot{v}$ is small compared to the external force $\boldsymbol{F}$, to zero order, the procedure is to use $m\dot{\boldsymbol{v}} = \boldsymbol{F}$ in the radiation reaction force, giving,

$$m\dot{\boldsymbol{V}} = \boldsymbol{F} + \tau_0\frac{d}{dt}\boldsymbol{F}, \tag{62}$$

which is the same as (15). It is also the same as the non-relativistic FO equation.

We may note, however, that this is only valid if, from (59), $\tau_0({}_1v^\sigma) << ({}_0v^\sigma)$. It has been shown, however, that this condition fails at optical frequencies as the intensity climbs above 10^{23} W cm^{-2}.[70] Intensities of 10^{22} W cm^{-2} have been reached by the Michigan group[71] and intensities of 10^{24} are targeted for the near future. Moreover, suppose v^σ in the third derivative term in (47) goes like $\cos\Omega t$. The condition that this term is small compared to the previous term is

$$\Omega << \sqrt{\frac{eE}{mc\tau_0}}. \tag{63}$$

This shows that for weak enough fields (63) may never be satisfied. Of course in this case the net force is extremely small, but for long times, such as charged particles in a galactic orbit, we see that we cannot even use the LL equation. Thus there is an entire range in which the LL equation seems to fail.

Another approach, writing $a^\mu = dv^\mu/d\tau$, rests upon an idea that arises by writing (46) as[70]

$$a^\mu - \tau_0\frac{da^\mu}{d\tau} = F^\mu/m \tag{64}$$

where F^μ is any external force. Since τ_0 is so small, the left hand side looks the Taylor series expansion of $a^\mu(t-\tau_0)$. In order to write all terms at the same time $t-\tau_0$, we also expand F^μ as

$$F^\mu(t-\tau_0) = F^\mu(t) - \tau_0 \dot{F}^\mu(t). \tag{65}$$

Noting that $\tau_0 \dot{F}(t) = \tau_0 \dot{F}(t-\tau_0) + \mathcal{O}\tau_0^2$, each term in (64) may be written at the same time, so that we have, using the electromagnetic force,

$$m\frac{dv^\mu}{d\tau} = \frac{e}{c}F^{\mu\sigma}v_\sigma + \tau_0 \frac{e}{m}\frac{d}{d\tau}(F^{\mu\sigma}v_\sigma). \tag{66}$$

As before this is easily patched to ensure that the identity $v_\sigma \dot{v}^\sigma = 0$ holds, and becomes

$$m\frac{dv^\mu}{d\tau} = \frac{e}{c}F^{\mu\sigma}v_\sigma + \frac{e\tau_0}{c}\left(\frac{d}{d\tau}(F^{\mu\sigma}v_\sigma) - \frac{v^\mu v_\gamma}{c^2}\frac{d}{d\tau}(F^{\gamma\nu}v_\nu)\right). \tag{67}$$

I derived this result not very long ago[72], but was mildly surprised to find that, to order τ_0 it had been derived years earlier by Ford and O'Connell[23]. In the non-relativistic limit, by the way, this is the same as the non-relativistic Landau Lifshitz equation (62). However, they used a completely different approach. They started from a generalized quantum Langevin equation and gave the electron structure. Then they had to assume a particular form factor, mass renormalization, and derive a result that is dependent on a finite cutoff parameter they call Ω. This parameter is removed from the equation in an iterative process, similar to the approach used in obtaining the LL equation.

To indicate (but not prove) the validity of (67) we may consider a simple problem. Suppose there is an electron initially at rest and a plane electromagnetic wave is incident upon it. We know that the light will impart momentum to the electron and it will accelerate parallel to the direction of propagation. Let us call I the intensity, or magnitude of the Poynting vector. The radiation pressure on the electron is equal to I/c so, using the Thomson cross section σ_T, we expect the force to be $I\sigma_T/c$, which produces an acceleration

$$a = \frac{I\sigma_T}{mc}. \tag{68}$$

However, it is well know that for a plane wave (23) *does not* give this result. This is an uncanny situation, particularly when we consider that the Lawson Woodward theorem also tells us that the electron should resist all net motion. How then, does the unsuspecting electron ever acquire momentum? The solution to this conundrum is that radiation reaction must be taken into account. In fact, when (67) is used as the radiation reaction force, I was able to show that (68) follows from the equation of motion.[72] There is moral to this little anecdote. It is an example that demonstrates, no matter how small the applied force is, one must take radiation reaction effects into account in order to correctly describe the true physical behavior.

Let as look at this issue from another point of view, and consider our assumptions carefully. The most fundamental assumption is embodied in (7), which is essentially conservation of energy. It would be dreadful to have to abandon this. However, there is a further unnamed assumption adopted in the ensuing steps, i.e., that the power radiated and the reaction force are evaluated at the same time. There are severals hints that may be harbingers pointing to the possibility that this assumption is overreaching.

Let us go back to the causality issue. When we allowed for violation of causality in the non-relativistic case, we obtained sensible solutions. We further saw that, with a "smoothed step function" causality was not, strictly speaking, violated. This case shows that the solution with radiation, for early times, acquires a given value (say $V = 0.5$ of Fig. (1)) *before* V_0 does, the solution without radiation reaction. We would expect just the opposite, but this is a residual effect of what we called pre-acceleration exhibited by the true step function force. The lesson of all this, it seems to me, is that we cannot be sure that the power radiated is to be evaluated at exactly the same time as the radiation force.

In the introduction we saw that runaway solutions are avoided if the electron has a non-zero size. In essence, the non-zero size allows for the fact that the electron, in fact, feels the force at different times than when it radiates, since it takes time for the wave to propagate across the finite distance. The time it takes light to traverse a classical electron radius a_0 is $a_0/c \sim \tau_0$, so we expect the self force and the radiated power may be evaluated at times differing by a lapse of the order of τ_0.

This idea may also be fleshed out without finite size effects. Although we are considering a classical model, this does mean we should follow Oedipus and blind ourselves, in this case, to quantum mechanics, which warns us not to prescribe the position, or time, with infinite precision. If we do, we find that the energy will be infinite, which is exactly what happens with the runaway solutions! But, by assuming that the self force is evaluated at *precisely* the same time as the power radiated, we are doing just that. In fact, quantum mechanics signals an uncertainty in time according to $\Delta E \Delta t \geq \hbar/2$. Using the electron rest energy, and taking the equality limit, this relation tells us that $\Delta t = \tau_0/\alpha$, where α is the fine structure constant.

So, for all of these reasons, suppose we entertain the notion that the self force, which we will now call R^σ, is evaluated at a different time, i.e.,

$$m\dot{v}^\sigma(\tau) = \frac{e}{c} F^{\sigma\mu}(\tau) v_\mu(\tau) + R^\sigma(\tau - \epsilon) \tag{69}$$

where ϵ is expected to be the order of τ_0. Equivalently, this equation may be written as

$$m\dot{v}^\sigma(\tau + \epsilon) = \frac{e}{c} F^{\sigma\mu}(\tau + \epsilon) v_\mu(\tau + \epsilon) + R^\sigma(\tau). \tag{70}$$

The idea is to explore the significance of this equation with ϵ taken as a parameter to be chosen to eliminate pathological behavior. Looking back at the steps leading to (66), we see that this is essentially what was going on there. To find R^σ we, as usual, equate the power radiated to minus work done by the electron to find $R^\sigma = m\tau_0 \ddot{v}^\sigma$. This is just

Equation of Motion	$\dot{v}^\mu = (e/mc)F^{\mu\sigma}v_\sigma + \mathcal{G}^\mu$
LAD	$\mathcal{G}^\mu = \tau_0\left(\ddot{v}^\mu + v^\mu \dot{v}_\sigma \dot{v}^\sigma/c^2\right)$
LL	$\mathcal{G}^\mu = \tau_0\left((e/mc)\dot{F}^{\mu\sigma}v_\sigma + (e/mc)^2(F^{\mu\gamma}F_\gamma{}^\phi v_\phi + F^{\nu\gamma}v_\gamma F_\nu{}^\phi v_\phi v^\mu)/c^2\right)$
FO	$\mathcal{G}^\mu = +(e\tau_0/mc)\left(\frac{d}{d\tau}(F^{\mu\sigma}v_\sigma) - v^\mu v_\gamma \frac{d}{d\tau}(F^{\gamma\nu}v_\nu)/c^2\right)$
MP	$\mathcal{G}^\mu = (e_1/c)F^{\mu\sigma}\dot{v}_\sigma + (2e^2/3m^2c^6)F^{\nu\sigma}\dot{v}_\nu v_\sigma v^\mu$
SW	$\mathcal{G}^n = -\tau_0\omega^2\gamma^4 v^n$
HL	$\mathcal{G}^n = -\tau_0\gamma^6\dot{v}^2 v^n/c^2$
Y	$\mathcal{G}^\mu = \theta(\tau)\tau_0\left(\ddot{v}^\mu + \frac{v^\mu}{c^2}\dot{v}_\sigma\dot{v}^\sigma\right)$
H	$\mathcal{G}^\mu = \phi^{,\mu} - v^\mu\dot{\phi}/c^2$

Table 1 A selected list of putative equations, including LAD, LL, FO, the Mo and Papas[15] equation, and the spatial part of the force for (a special case of) Steiger and Woods,[18] Hartemann and Luhman,[27] Yaghjian, and the author.

what was done for the LAD derivation, but there we did evaluate all things at the same time. Expanding all terms involving $(\tau+\epsilon)$ in a Taylor series about τ, we find,

$$m\dot{v}^\sigma + m\epsilon\ddot{v}^\sigma = eF^{\sigma\mu}v_\mu + \epsilon e\frac{d}{d\tau}F^{\sigma\mu}v_\mu + m\tau_0\ddot{v}^\sigma \tag{71}$$

to order τ_0. Now, we know the troublemaker is $\ddot{v}^\sigma$, so it is sensible to take $\epsilon = \tau_0$. The resulting equation is not quite right, suffering from the previous complication that it does not satisfy $v_\sigma\dot{v}^\sigma = 0$, so we add the corrective term as before. This gives

$$\frac{dv^\sigma}{d\tau} = F^{\sigma\mu}v_\mu + \tau_0 v^\sigma \dot{v}_\mu \dot{v}^\mu - \tau_0\frac{d}{d\tau}(F^{\sigma\mu}v_\mu), \tag{72}$$

which is exactly the same as (66).

Another approach was developed by Yaghjian.[73] He modeled a particle by a shell. He assumed that no forces act upon the shell until the time $t = 0$, when the force is applied, and obtains

$$\frac{dv^\mu}{d\tau} = \frac{e}{mc}F^{\mu\sigma}v_\sigma + \theta(\tau)\tau_0\left(\ddot{v}^\mu + \frac{v^\mu}{c^2}\dot{v}_\sigma\dot{v}^\sigma\right). \tag{73}$$

Due to the presence of the step function $\theta(t)$, this obviously avoids preacceleration. This approach, called LADY, was studied by Eriksen and Grøn.[74] This is a good time to summarize the relativistic equations of motion we have so far, collected in Table 1.

The LAD equation seems to be derived from rock-solid principles–relativity and energy conservation–yet is quite unacceptable. The runaway solutions quash any remnants of reality, leaving an abiding yet chimerical equation. The LL equation appeared in their book[12] without much justification. The derivation above outlines limits of its usefulness, and we see that it fails for high enough energy. Nevertheless, as far as radiation reaction is concerned, it is probably the most used equation of the 20th century. One may also

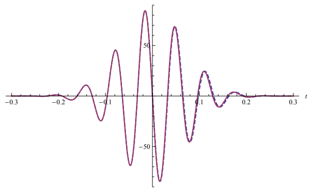

Fig. 11 A plot of ${}_0v^1$ (solid, no self force) and v^1 (dashed includes self force) at $I = 10^{23}$ W cm^{-2} and...

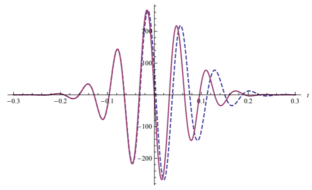

Fig. 12 $I = 10^{24}$ W cm^{-2}

note this can be put in many equivalent forms. This is achieved by using (23) in the LL equation and keeping terms to order τ_0. There is no unique form to this order. On the other hand, however, a while ago Rohrlich[63] claimed that the LL is in fact exact. He argued in favor of a derivation based on the approach that looks at the force on an extended electron in the limit that the radius goes to zero. He stated it was rigorous based on the work of Spohn.[29] However, Spohn used a series approach and cuts it off at first order in the small parameter ϵ, noting that there are higher order terms that are neglected.

Let us look at a concrete example. We consider a pulse of intensity $I = 10^{23}$ W cm^{-2} and $I = 10^{24}$ W cm^{-2} of the form (36) with $w = 0.1$ and $\Omega = 50$. This corresponds to a pulse in the ultraviolet a few wavelengths in length. To analyze the effects of self forces, let us use the LL (60) equation and compare it to the equation of motion without radiation reaction. We shall compare the z component of the velocity for each case, calling ${}_0v^3$ the solution without self forces and v^3 the solution with self forces, the solution of (60) to order τ_0, which is found numerically (using Mathematica). The results are displayed in Figs. 11 and 12. The graphs show that, by the end of the pulse, the results are substantially different at $I = 10^{24}$ W cm^{-2}, but not so different at $I = 10^{23}$ W cm^{-2}.

This is a concrete example showing that radiation reaction becomes important for intensities over $I = 10^{23}$ W cm^{-2}. Of course, it may be important for much lower intensities, as was seen above, but for the extreme light experiments that are about to reach and surpass this threshold, we see that self forces are essential to the analysis.

Two other points are that the difference in the velocity between ${}_0v^1$ and v^1 can be quite measurable, since, if the particle is thrown free, it may change the direction by 180 degrees! Many other effects, including measurements of the radiation field, will be measurable. The other point is that this also shows that the LL equation begins to fail just when it is most needed (in this regime). Its derivation was based on the smallness of $\tau_{0\,1}v$ compared to ${}_0v$. Fig. 12 shows that this is no longer true!

The FO equation is a new player, but I have shown it passes at least one test, described above. I presented two derivations that are very simple and heuristic, but the original derivation is due to Ford and O'Connell.[23] The issues associated with this are discussed above, and it too is not exact.

Thus, it seems we have a number of choices, the LL and the FO equation, plus the others mentioned in the introduction and those in the Table. However, to order τ_0 FO and LL are the same, as I will show. Thus, since each one appears to be valid only to order τ_0, it does not matter which one we use. The main trouble is what we should do in the extremely high field limit. I will discuss this more later.

To see the equivalence of LL and FO to order τ_0, let us write the LL equation as

$$\left(\frac{dv^\mu}{d\tau}\right)_{\text{LL}} = F^\mu_{\text{LL}}, \tag{74}$$

where the right side of (74) is given by comparison to (60). Similarly we have

$$\left(\frac{dv^\mu}{d\tau}\right)_{\text{FO}} = F^\mu_{\text{FO}}, \tag{75}$$

where the right side of (75) is given by comparison to (67). Now let us consider the difference, Z^μ, defined as

$$Z^\mu = \left(\frac{dv^\mu}{d\tau}\right)_{\text{LL}} - \left(\frac{dv^\mu}{d\tau}\right)_{\text{FO}}. \tag{76}$$

We find

$$\frac{Z^\mu}{(e/mc)} = \tau_0(e/mc)v^\mu \left(F^{\sigma\lambda}v_\lambda F_\sigma{}^\nu v_\nu - v_\sigma \frac{d}{d\tau}(F^{\sigma\nu}v_\nu)\right). \tag{77}$$

Since Z^μ is a quantity of order τ_0, (23) may be used in (77) to obtain

$$\frac{Z^\mu}{(e/mc)} = \tau_0(e/mc)v^\mu\left(\dot{v}^\sigma\dot{v}_\sigma - v_\sigma\ddot{v}^\sigma\right) = 0. \tag{78}$$

This shows that LL and FO are the equivalent, to order τ_0.

4.2 The constant force

Half a century ago Thomas Fulton and Fritz Rohrlich wrote, "The old and much-debated question, whether a charge in uniform acceleration radiates, is discussed in detail and its implications are pointed out."[46] One of the most perplexing problems in electrodynamics, this has been nagging generations of physicists for a century. It was unresolved 50 years ago, and little progress has been made since. The sentence given voice by Fulton and Rohrlich could be used today just as well as when it was so many years ago.

It can be seen right from the start that the LAD equation is in trouble when constant fields are under consideration. To see this, let us integrate the time component of the LAD equation (47) with respect to proper time. This gives,

$$mc^2(\gamma - \gamma_{\text{inc}}) = \int \boldsymbol{F}\cdot d\boldsymbol{x} - \int P dt + \tau_0(\dot{v}^0 - \dot{v}^0_{\text{inc}}). \tag{79}$$

where $\boldsymbol{F} = e\boldsymbol{E}$. We should emphasize that the electric field used in the LAD is what we measure in the lab, so we are using lab based coordinates. Thus, we measure x as length and t as time, which are what appear in the above. The physical interpretation of (79) is easy to see: it reads, the change in kinetic energy is equal to the work done by the external field minus the energy radiated away *plus something else*. The *something else*, which is the Schott energy, destroys our concept of what conservation energy should be, so we see that the LAD equation must lead to very odd behavior indeed. However, we should note that $\dot{v}^0 = \boldsymbol{U} \cdot \dot{\boldsymbol{v}}$. This term, evaluated at the initial and final times, vanishes if the initial and final acceleration is zero, but not in general, and not for a uniform constant electric field, and not for an electron trapped in a magnetic field. Thus, the LAD equation will not give sensible (i.e., energy conserved) results for these cases. One say generalize and say, since the LAD equation fails for these cases, it is not trustworthy in general, but as we have seen, there are bigger problems with this equation. It is emphasized that the *something else* arises from the Schott term, the term that leads to the unacceptable behavior documented above.

Let us begin by assuming that there is a constant, uniform electric field in the lab frame, oriented so that the field is in the x direction. With this, ignoring radiation reaction initially, (23) gives

$$\dot{v}^0 = \frac{g}{c} v^1 \tag{80}$$

and

$$\dot{v}^1 = \frac{g}{c} v^0 \tag{81}$$

where $g = eE/m$. The solutions to these equations are easy to find and given by the well-known “hyperbolic motion” equations:

$$\frac{v^0}{c} = \cosh(g\tau/c) \tag{82}$$

and

$$\frac{v^1}{c} = \sinh(g\tau/c), \tag{83}$$

assuming the initial velocity is zero.

These are sensible solutions showing that the four velocity is unbounded, as we expect with a constant force. Of course, τ is the proper time. Remembering that $v^1 = dx/d\tau = V(dt/d\tau)$, we may transform the results to lab measured values V and t,

$$V = \frac{gt}{\sqrt{1 + (gt/c)^2}}. \tag{84}$$

So far everything seems ordinary. The only complaint one might raise is that a truly constant field is unphysical. While this is true, for small regions we may obtain very nearly constant fields, so these results should approximate reality in a limited region.

The first issue, that surfaces from time to time, is the inappropriate application of the principle of equivalence, which says that in a small spacetime region the acceleration is like a gravitational field. The smaller the region, the better the approximation. One might say that at a point the gravitational field is equivalent to an accelerated frame, but this is no more than collection of words with little meaning. This is because observables, such as speed or acceleration, require measurements at two different points (in space or time or both).

After he developed his special theory of relativity in 1905, Einstein spent some time trying to generalize that theory to account for gravitational fields. In 1907 he hit upon the principle of equivalence, and was able to predict that gravity would affect the path of a light ray, which was subsequently proved in the eclipse of 1919. However, in 1907 Einstein did not know about Riemannian geometry, the geometry of curved spacetime. His friend, Marcell Grossman taught Einstein the mathematics that describe this geometry. Eventually Einstein did not need the principle of equivalence: It was learned that the curvature of space is determined by the Riemann tensor, who introduced it in 1845 in his habilitation lecture, the qualifying exam (in those days) to lecture at a university.

Today we have an unequivocal meaning of the Riemann curvature tensor: If it is zero then space is flat, if it is not zero than space is curved. The curvature of space replaces the Newtonian concept of a gravitation field. For a uniformly accelerated frame (in fact, for any acceleration in Minkowski spacetime) the curvature tensor vanishes: There is no gravitational field. The only nagging problem is that of an infinite planar source, which would be expected to give a uniform field, but such a source is unphysical, and so one must look at a realistic source and consider, if one is so inclined, to very short distance or times. For a somewhat recent discussion of the equivalence principle with some references, one may see [75].

The problem, that arises from time to time, is arguing that an electron falling through a constant field, suffering an acceleration, is equivalent to an electron at rest in a gravitational field. This is the paradox: The accelerating electron radiates while the "equivalent" electron at rest does not radiate energy. The resolution of the paradox is explained above, and the accelerating electron radiates while the electron at rest does not. This issue has a long and lively history, and various investigations look into this question in great detail. One may consult [65] for a deeper discussion on this issue.

However, there is a greater problem than this facing the hapless electron in a uniform field. First, let us calculate the radiated power, P, from (44). It yields,

$$P = m\tau_0 g^2, \tag{85}$$

a sensible result, telling us that the electron radiates power at a constant rate.[7] But a storm of controversy arose over these results–the first clouds were embodied by the LAD equation, (47). Using the identity $\cosh^2 x - \sinh^2 x = 1$, one may show by direction substitution that (82) and (83) satisfy (47)! Thus, while the electron happily radiates energy, its motion is unaffected by this radiation. Such a situation clearly violates how we expect conservation of energy to work. As it radiates energy, the electron must sacrifice some its kinetic energy to radiation, yet it does not.

Eriksen and Grøn[65] put it thus: "Consider for example a freely falling charge moving vertically along a geodesic world line. In this case there is no radiation reaction. Hence a neutron and proton falling vertically besides each other will proceed to move together.

[7] Interestingly, this is the same result one finds using the non-relativistic equations.

Yet the proton radiates, but not the neutron. Where does the radiated energy come from?" They proceed to show that the answer lies in the Schott energy. Another possible solution that has been discussed here and there[46] is that the *internal energy* of the electron changes. This idea was also embraced by Teitelboim,[11] who shows that the Schott term is related to the momentum of the particle, and does not appear in the far field as radiation. Taking this one step further, van Weert introduces a third rank tensor, the divergence of which is related to this momentum.[76] The two approaches, integrating along the world tube and using the equation of motion for a charged particle, have recently been compared.[77] Since all electrons (including those, presumably, that have radiated for countless eons as they are accelerated willy-nilly through their cosmic history) are measured to have the same mass, this idea seems untenable. However, Bonnor[78] uses (58) for macroscopic sized objects, in which the mass can change due to a modification in the internal structure of the object. He warns that this is only valid for macroscopic objects, "...this theory is not intended to describe electrons. I suppose that it is applied to macroscopic particles up to a certain stage, as yet unknown, in the process of mass decay, and after that some other theory should replace it." Bonnor also credits Larmor as being the originator of this idea in a lecture he gave in 1912.

The other solution is to argue that the LAD equation is wrong. This was a bit too revolutionary, it appears, at the time Fulton and Rohrlich published their paper, but nowadays a new regime is in force, and these words no longer provoke the winds of war. Unfortunately we get equally egregious behavior when the LL equation is used to order τ_0. In particular, if one uses (82) and (83) in the terms in (60) that are multiplied by τ_0, we find that that entire term vanishes, so that (82) and (83) is a solution to LL, to order τ_0. This peculiar result has been noted often, and recently.[51] One might argue that there is not really a problem at all, one need only go to order τ_0^2 to see the effect. However, this is not the case with the LAD equation: Moreover, consider the following. Suppose there is an electric field that only slightly differs from a uniform field, this difference may be characterized by some parameter ϵ. As long as $\epsilon > 0$, we have radiation, to order τ_0. It is unreasonable to expect in the smooth limit that $\epsilon \to 0$, the order τ_0 radiation suddenly jumps to order τ_0^2 radiation.

In addition, the FO equation suffers the same ailment. This is no surprise according to the derivations above, but the original derivation eschewed the LAD equation as a starting point. If one assumes that an acceptable equation of motion must be devoid of the above contradiction, then FO too falls from grace.

To see this, we may again use (82) and (83) in (75). It is necessary to use $v_\sigma v^\sigma = c^2$ and $\dot{v}_\sigma v^\sigma = 0$. With that, once again we run aground on that same spit of land where a particle radiates energy without suffering the slightest effects from that radiation. The law of conservation energy seems outlawed on this strange island paradise.

5. Another approach

None of the existing equations of motion is generally accepted as exact and true. Although each has is followers and proponents, all are based on some sort of assumptions and/or expansions. Let us now suppose that we make no new assumptions, other than, a) conservation of energy and, b) there is a radiation reaction force f. For the non-relativistic case in one dimension we have

$$m\dot{V} = F - f \tag{86}$$

as before, except we now *change the sign of* f. Conservation of energy gives, again,

$$fV = m\tau_0(\dot{V})^2. \tag{87}$$

If integrated from t_1 to t_2, then (87) is a statement of conservation of energy, stating that the work done by the radiation reaction force, f, is the energy radiated. The trouble, seen throughout the entire proceeding material, comes after performing the integration, which gives rise to the Schott term and turns us down an avenue of pitfalls and troubles.

Let us avoid this old path of dead ends and bizarre results, and consider the possibility that we stop here, following where the physics takes us. Thus (in one dimension) we have two equations, (86) and (87), two unknowns the velocity and the radiation force. The force f can be formally eliminated by multiplying by V and using (44)

$$mV\dot{V} = VF - m\tau_0\dot{V}^2. \tag{88}$$

Integrating this nonlinear equation gives exactly what we expect,

$$\frac{1}{2}mV^2 = \int Fdx - \int Pdt, \tag{89}$$

or, the kinetic energy equals the work done by the external force minus the energy radiated away. The good news is that (88) is exact and free of the plagues of the conventional approach. The bad news is that, in general, exact solutions are difficult to find.

Thus, the radiation reaction force is never found directly, nor need it be, since it is the motion we measure, not the force. However, to relate to the notion of a radiation reaction force we solve (87) for f, i.e., $f = m\tau_0(\dot{V})^2/V$. This must be generalized to three dimensions, which is done below, and care must be taken to avoid singularities. The relativistic approach is given elsewhere.[20]

An illustrative way to look at this problem is to consider a very special kind of motion, one such that the acceleration is given. Essentially this already gives us the solution (which does not happen generally) but this nevertheless will be a useful exercise. Let us suppose a charged mass is accelerated such that $a = a_0 e^{-(t/T)^2}$, but in the graph we take a_0 and T to be unity. The power radiated is $m\tau_0 a^2$ and

$$f = \frac{2ma_0\tau_0 e^{-2(t/T)^2}}{\sqrt{\pi}T(1 + \mathrm{erf}(t))}, \tag{90}$$

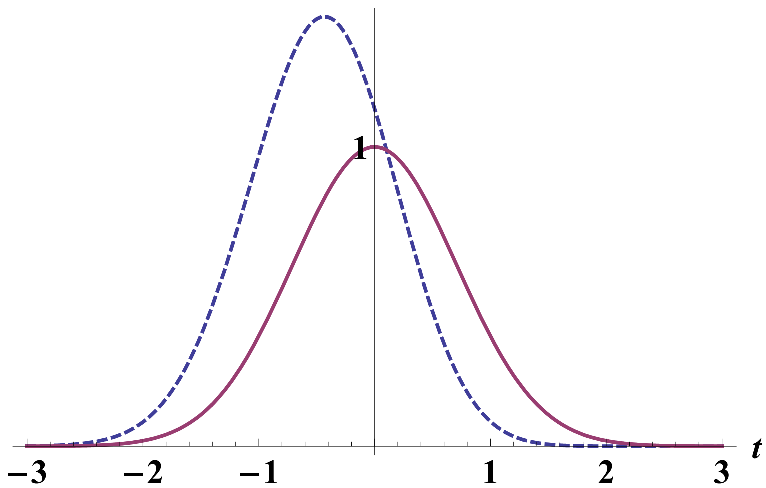

Fig. 13 The acceleration divided by m and the radiation reaction force divided by $m\tau_0$. The self force peaks earlier than the acceleration.

assuming $v(-\infty) = 0$.

It is interesting to see that the radiation reaction force peaks before the acceleration, a result reminiscent of the pre-acceleration solutions. In fact, although the third order term is nowhere in the formulation, this graph shows a remnant trace of that specter. I refer to the fact that the derivative of a Gaussian (which would correspond to the $\dot{a}$ term) peaks earlier than the Gaussian.

Before moving on to the relativistic generalization, it is helpful to look at two other simple classes of problems. For the first we assume that the external force is of the form $F = bt^n$ and that we can expand the velocity as

$$V = {}_0V + \tau_0({}_1V) \tag{91}$$

Elementary calculations give

$$V = \frac{b}{m}\left[\frac{t^{n+1}}{n+1} - \tau_0\frac{n+1}{n}t^n\right] \tag{92}$$

which is fine for $n > 0$. In fact, as a check, one easily finds that the kinetic energy is, letting $\Upsilon = b^2/m$,

$$\mathrm{KE} = \Upsilon\left[\frac{t^{2n+2}}{2(n+1)^2} - \frac{\tau_0}{n}t^{2n+1}\right], \tag{93}$$

and the energy radiated away, W_R, is

$$W_R = \tau_0\Upsilon\frac{t^{2n+1}}{2n+1} \tag{94}$$

and the work done by the external force, $W_F = \int F({}_0V + \tau_0({}_0V)dt$ is

$$W_F = \Upsilon\left[\frac{t^{2n+2}}{2(n+1)^2} - \tau_0\frac{(n+1)t^{2n+1}}{n(2n+2)}\right] \tag{95}$$

This shows that (for $n > 0$), to order τ_0, the kinetic energy equals the work done by the external force minus the energy radiated away, a necessary check on the correctness of the radiation reaction force f.

For the case that $n = 0$ we find that f becomes singular and therefore, since ${}_1u$ also becomes singular, the expansion (91) fails. This, however, is a very important special case, it is the constant force problem, which we shall look at below.

Another simple yet important example is the constant magnetic field, although we must generalize to three dimensions. To obtain a sensible result we divide (88) by V, assuming the resulting quotient is well behaved. Generalizing the result to three dimensions we have

$$\boldsymbol{f} = m\tau_0 \frac{a^2}{V^2} \boldsymbol{V}. \tag{96}$$

In this, it is assumed that the radiation reaction force $\boldsymbol{f}$ is antiparallel to the velocity. There is ample precedent for taking this force to be antiparallel to the velocity, see Refs. [19] and [27]. To show this makes sense, we consider yet another simple problem: A charged mass rotating in a circle of radius R, confined by a circular collar that provides the inward normal force. We assume it has an initial velocity V_0 and initial angular velocity ω. Once again, to order τ_0, the analysis is elementary and we find that $f = m\tau_0\omega^3 R$ and $v = v_0 - m\tau_o\omega^3 Rt$. This yields for example, the change in kinetic energy per period T is,

$$\frac{\Delta KE}{T} = m\tau_0\omega^2 R^4 \tag{97}$$

The right hand side is exactly what the Larmor formula predicts.

For the relativistic approach, I will follow the work of Ref. [20]. It is assumed that, corresponding to the power *scalar*, there is a scalar ϕ that is related to the energy radiated and a force derived according to $f_\sigma = \phi_{,\sigma}$. With this we have -

$$m\frac{dv^\mu}{d\tau} = \frac{e}{c}F^{\mu\sigma}v_\sigma + \phi^{,\mu} - \frac{v^\mu}{c^2}\dot{\phi} \tag{98}$$

where the last term is added to maintain the identity $v_\sigma \dot{v}^\sigma = 0$, and we identify $-f^\mu = \phi^{,\mu} - \frac{v^\mu}{c^2}\dot{\phi}$ as the self force.

If we integrate (98) with respect to proper time we find,

$$mc^2(\gamma - \gamma_{\text{inc}}) = \int \boldsymbol{F} \cdot d\boldsymbol{x} - c\int f^0 d\tau. \tag{99}$$

Conservation of energy implies that

$$f^0 = \gamma P/c. \tag{100}$$

Thus, (98) with (100) gives a complete solution to the self force problem.

Since τ_0 is so small, it is sometimes useful to consider the series,

$$v^\sigma = {}_0v^\sigma + \tau_0({}_1v^\sigma). \tag{101}$$

Ref. [20] gives the solution to the constant electric field, and for the constant magnetic field, assuming γ is slowly changing and therefore constant to $\mathcal{O}(\tau_0)$, it was shown

$$v^1 = u\cos\omega\tau(1 - b\tau) \tag{102}$$

$$v^2 = -u\sin\omega\tau(1 - b\tau) \tag{103}$$

where $b = \tau_0\omega^2(1 + u^2/c^2)$ and u is the initial velocity. These were integrated to find the position as a function of proper time and are plotted in Fig. 14.

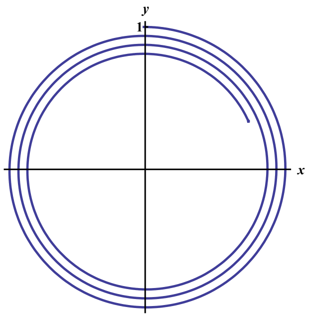

Fig. 14 Parametric plot of x and y versus proper time, showing the electron spiraling in due to radiation reaction. For illustrative purposes, I set $u = 1$, ω=1, and $b = 0.01$ (which, of course, corresponds to a huge and false value of τ_0).

6. Summary

The need for the correct equation of motion with radiation reaction is established due to the extremely high intensities of current lasers, and the higher intensities written into current proposals. The major contenders, the LAD (Lorentz Abraham Dirac), the LL (Landau Lifshitz), and the Ford O'Connell (FO) were discussed, although the LAD is usually abandoned at the outset due to the runaway solutions. These and other equations were collected in the Table.

A section on relativistic equations of motion without radiation investigated the electron motion in a sinusoidal field and a short pulse. That section presents results originally found in the 1970s and later, but collect some of the salient features of the problem and cements the notation.

Derivations were given for the LL and the FO equations that were based on an asymptotic expansion of the velocity. The results are not expected to be valid for extreme intensities, although, as explained in detail above, certain authors argue that their equation is, in fact, exact. Time will tell, and the exciting thing is, that time may be coming soon.

A brief history of the problem is given, and I add my own approach that avoids the unphysical behavior of the self force issue.

Acknowledgment

I would like to thank Cathy Mako of the Army Research Lab library for her invaluable efforts in obtaining articles and translations, and the referee for his careful reading and added references which greatly improved the reading and physics of this paper.

7. Appendix

This is a brief overview of electrodynamics in curved spacetime, or in the presence of a gravitational field. For details one may consult the literature.[12] The metric tensor $g_{\mu\nu}$ generalizes from the Minkowski metric (20). If the metric tensor can be derived from the Minkowski metric through a coordinate transformation, then the space (short for spacetime) is flat, if it cannot be derived from a coordinate transformation from the Minkowski metric, then the space is curved. The source of curvature is mass or energy.

Many times, the effects of gravitation can be deduced from the "comma to semicolon" rule, i.e.,

$$A_{\mu,\nu} \to A_{\mu;\nu} \tag{104}$$

where the so-called covariant derivative is defined by

$$A_{\mu;\nu} = A_{\mu,\nu} - \{{}^{\,\sigma}_{\nu\mu}\} A_\sigma \tag{105}$$

where the Christoffel symbol is defined as

$$\{{}^{\,\sigma}_{\nu\mu}\} = \frac{1}{2} g^{\sigma\gamma} \left(g_{\gamma\mu,\nu} + g_{\gamma\nu,\mu} - g_{\mu\nu,\gamma} \right). \tag{106}$$

This is how the effects of a gravitational field manifest themselves. If space is curved (if a gravitational field is present), then the Christoffel symbols do not vanish. However, the converse of this statement is not true. Even in flat spacetime, curvilinear coordinates will give non-zero Christoffel symbols. The unequivocal test for the existence of the gravitational field is through the curvature tensor,

$$R_{\beta\mu\nu}{}^{\sigma} = \{{}^{\,\sigma}_{\mu\nu}\}_{,\beta} - \{{}^{\,\sigma}_{\beta\nu}\}_{,\mu} + \{{}^{\,\sigma}_{\beta\phi}\}\{{}^{\,\phi}_{\mu\nu}\} - \{{}^{\,\phi}_{\mu\phi}\}\{{}^{\,\sigma}_{\beta\nu}\}. \tag{107}$$

The homogeneous Maxwell equations turn out to be unaffected but the other equations are given by

$$F^{\nu\mu}{}_{;\nu} = 4\pi j^\mu \tag{108}$$

where

$$j^\mu = \frac{e}{\sqrt{-g}} \int \delta\left(x - x(\tau)\right) v^\mu \tag{109}$$

where g is the determinant of the metric tensor and

$$F^{\nu\mu}{}_{;\nu} = F^{\nu\mu}{}_{,\nu} + \{{}^{\,\mu}_{\nu\phi}\} F^{\phi\nu} + \{{}^{\,\nu}_{\nu\phi}\} F^{\mu\phi}. \tag{110}$$

Since $F^{\nu\mu}$ is antisymmetric while $\{{}^{\mu}_{\nu\phi}\}$ is symmetric (in its lower indices), the second term on the right is zero, but due to the antisymmetry of the $F^{\nu\mu}$ and the identity $\sqrt{-g}_{,\sigma}/\sqrt{-g} = \{{}^{\nu}_{\nu\sigma}\}$, (108) may be written as

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{-g} F^{\nu\mu}\right) = 4\pi j^{\nu}. \tag{111}$$

The equations of motion of a charged massive particle are

$$\frac{Dv^{\sigma}}{D\tau} = \frac{e}{m} F^{\sigma\mu} v_{\mu} \tag{112}$$

where $\frac{Dv^{\sigma}}{D\tau}$ is the covariant derivative along the curve, defined as

$$\frac{Dv^{\sigma}}{D\tau} = \frac{dv^{\sigma}}{d\tau} + v^{\mu} v^{\nu} \{{}^{\sigma}_{\mu\nu}\}. \tag{113}$$

The point of all of this is to show that when the curvature tensor vanishes, and assuming we adopt Cartesian coordinates, the Christoffel symbols vanish and there is no effect of gravitation on the charged particle. If we use another coordinate system (other than Cartesian), then the Christoffel symbols may not vanish, but clearly the physics is unchanged. This shows that if one transforms to an accelerated reference frame from flat space then space is not curved. Thus, a charged particle at rest at the surface of the Earth (ignoring rotation, etc.) is not equivalent to a particle in a uniformly accelerated frame. The particle at rest on the surface of the earth is in gravitational field (with non-zero Christoffel symbols), but since it is not accelerating it does not radiate. A particle suffering uniform acceleration does radiate, but a uniformly accelerated reference frame (obtained via a coordinate transformation from flat space) is not equivalent to a gravitational field, since the curvature tensor vanishes.

The principle of equivalence was very important to Einstein in helping him develop his equations of general relativity, but once the physical significance of the Riemann tensor was understood, and the equations of motion derived, there was little room left for it. The relativist Synge put it best: "The Principle of Equivalence performed the essential office of midwife at the birth of general relativity. . . . I suggest that the midwife be now buried with appropriate honours... ."

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