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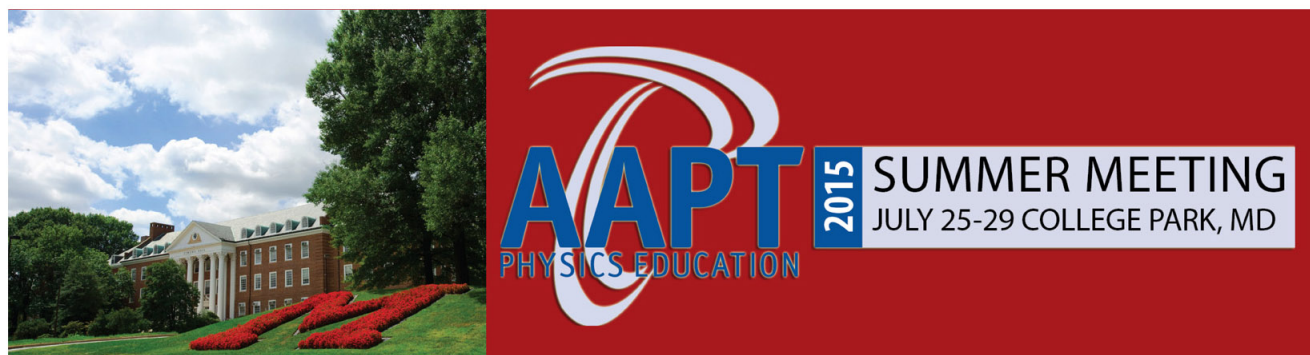
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# Reduced-order Abraham-Lorentz-Dirac equation and the consistency of classical electromagnetism

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It is widely believed that classical electromagnetism is either unphysical or inconsistent, owing to pathological behavior when self-force and radiation reaction are non-negligible. We argue that there is no inconsistency as long as it is recognized that certain types of charge distribution are simply impossible, such as, for example, a point particle with finite charge and finite inertia. This is owing to the fact that negative inertial mass is an unphysical concept in classical physics. It remains useful to obtain an equation of motion for small charged objects that describes their motion to good approximation without requiring knowledge of the charge distribution within the object. We give a simple method to achieve this, leading to a reduced-order form of the Abraham-Lorentz-Dirac equation, essentially as proposed by Eliezer, Landau, and Lifshitz and derived by Ford and O'Connell. © 2015 American Association of Physics Teachers.

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## I. INTRODUCTION

This paper addresses two separate but related issues. One issue is the correct understanding of pathological behavior exhibited by certain equations of motion when self-force is non-negligible. The second issue is the formulation of a valid equation of motion for a small charged body that does not exhibit pathological behavior. In both cases, the aim of the present paper is to provide arguments that are rigorous but simple, for results that have previously been obtained by more sophisticated methods.

It is a well-known feature of classical electromagnetism that any full treatment of the motion of a charged object includes some subtleties surrounding the issue of self-force and radiation reaction. One can use the theory to obtain an equation that describes, approximately, the motion of a rigid spherical shell of charge subject to an arbitrary force. This is the Abraham-Lorentz-Dirac (ALD) equation.<sup>1-6</sup> This equation, and others like it, is problematic because it is higher than second order in the time derivatives of position, which can lead to problems with causality, and because it has pathological or “runaway” solutions in which, for example, an object accelerates when no force is applied.

It is well established that the pathological cases can be ruled out by the following strategy. First, one replaces the ALD equation by a better approximation to the exact equation of motion of a charged spherical shell [Eq. (3.7) of Ref 4], called the “Caldirola equation”; see also Refs. 3, 7, and 8, and then one insists that an entity of given charge and observed mass cannot have a radius below a certain minimum. This both ensures the absence of pathological behavior<sup>4,8-11</sup> and is physically to be expected because it is the condition that the observed mass must exceed the electromagnetic contribution (as we will expound further in the following). However, there is continuing argument about what is the right way to interpret this situation<sup>4,7,11-13</sup> and there is resurgent interest in this whole area because modern laser technology makes it possible to experimentally investigate radiation reaction phenomena.<sup>14</sup>

We will here put forward the point of view, also espoused by several earlier authors, that, if gravitational effects are negligible, then *a point-like particle with finite charge and finite observed mass is a strict impossibility in classical*

*physics.*<sup>13,15-17</sup> More generally, we will argue that the issue is that there cannot exist an entity whose inertial mass is negative, and one should not model any physical system as if it was equivalent to one containing an entity whose inertial mass is negative. We are not the first to espouse this point of view, and indeed the fact that negative bare mass leads to instability has long been recognized;<sup>18</sup> for reviews, see Refs. 16, 19, and 20. We re-assert and expound it a little further.

If, in the absence of gravitational effects, there cannot be any such thing as a point particle with finite charge and mass in classical physics, then there is no exact or correct equation of motion for such an entity in flat spacetime. The search for such an equation, or for approximations to it, must be abandoned.

The combination of gravitational and electromagnetic self-energy was considered by Arnowitt *et al.*<sup>21,22</sup> They show that, for a certain specific (and very small) charge-to-mass ratio, the divergent terms in the self-energy cancel, so that the point limit can be taken. However, such an approach does not allow a point-like model of electrons, whose charge-to-mass ratio is much higher; this will be briefly examined in Sec. II C.

It remains useful to identify equations of motion for small charged entities. One strategy to avoid problems is to adopt the ALD equation but add further boundary conditions that are chosen in such a way as to rule out the runaway solutions in practice.<sup>7,19,23,24</sup> Another is to develop new equations of motion<sup>17,25</sup> or to replace the ALD equation by a related equation such as the one commonly called the “Landau-Lifshitz equation.”<sup>11,14,26-29</sup> The Landau-Lifshitz equation is a reduced-order form of the ALD equation; this strategy is essentially the one independently described by Eliezer in 1948 (Ref. 26) and arguably should bear his name, so we shall refer to it as Eliezer-Landau-Lifshitz (ELL). (The precise relationship between the Landau-Lifshitz equation and the one proposed by Eliezer is presented in the appendix.)

The relationship of the work of Ford and O'Connell to that of Eliezer and Landau and Lifshitz, and other approaches, is discussed in recent reviews.<sup>14,30</sup> Landau and Lifshitz did not so much derive their equation as give arguments to suggest it was reasonable. Eliezer (independently<sup>31</sup>) went somewhat further by explicitly treating an extended charge distribution, and noting that there could possibly exist

a distribution such that the equation of motion takes the form he proposed [Eq. (A1) or (A5), or, slightly more generally, Eq. (29)]. Ford and O'Connell (FO)<sup>17,25,32</sup> derived the same equation, in the low-velocity limit, for a specific rigid charge distribution (one whose form factor is Lorentzian) treated in a dipole approximation, which is valid at low frequencies. Within those approximations, their result is exact for that charge distribution, and since they explicitly derived it the equation might fairly be attributed to them. Their method also extends to general charge distributions. In the following, I shall refer to the Landau-Lifshitz and Eliezer-Ford-O'Connell equations together as "ELL/FO," since for a general charge distribution and at arbitrary velocities and accelerations they are both approximate and differ only at the next higher order in a series expansion in powers of the characteristic distance or time (the width of the charge distribution or the time taken by light to traverse it). In any case, the form recommended by Landau and Lifshitz can be understood as a deliberate modification, merely for calculational convenience, of the equation of Eliezer and of Ford and O'Connell (see appendix).

Ford and O'Connell developed their treatment from first principles, starting from quantum electrodynamics. They obtained a quantum Langevin equation, from which a classical equation of motion can be derived. An important feature of their work is to show that the radiation reaction force is accompanied by fluctuations, which renders some other approaches (such as Dirac's) inadmissible because the use of the advanced Green function leads to a situation in which the fluctuations drive an instability. The calculation of Gralla *et al.*, by contrast, remains valid, as does the one presented here in Sec. III. Ford and O'Connell's approach can also be applied directly to the classical problem by the use of Poisson brackets instead of Heisenberg equations of motion;<sup>30</sup> therefore, it also offers insight into the purely classical problem, and this is what one is interested in when assessing whether or not classical physics is internally consistent.

Recently, there have been different approaches that explore the relationship between the ALD and ELL/FO equations. Medina<sup>10</sup> treats the exact equation of motion formally by writing down integral expressions for the fields obtained from potentials in the Coulomb gauge and then expanding the integrand in a power series, thus obtaining the ELL/FO equation. Gralla *et al.*<sup>13</sup> start out from energy-momentum conservation for continuous distributions of charge, via the stress-energy tensor. They reproduce the ALD equation and also obtain the ELL/FO equation by a suitably constructed series expansion while avoiding unphysical cases such as infinite field energy. Griffiths *et al.*<sup>11</sup> take as their starting point the Caldirola equation and obtain from it both the ALD and ELL/FO equations as equally legitimate approximations. In the present work, we justify the use of the ELL/FO equation by a straightforward argument inspired by, but much simpler than, the approach taken by Gralla *et al.*

## II. AVOIDING UNPHYSICAL CASES

First, we consider the problem of runaway solutions to equations of motion. We will argue that there is no known case of a runaway prediction when the motion is treated using standard electromagnetism (Maxwell's equations and the Lorentz force equation), as long as the equations are

handled correctly. We begin with the ALD equation and then consider the exact equation of motion for an extended charged body.

The ALD equation describes the motion, to first approximation, of a charged spherical shell under an applied external force. It can be written [taking  $c = 1$  and metric signature  $(-, +, +, +)$ ]:

$$\mathbf{F}_{\text{ext}} + \frac{2}{3}q^2 \left( -\frac{\dot{\mathbf{v}}}{R} + \ddot{\mathbf{v}} - \dot{\mathbf{v}}^2 \mathbf{v} \right) + O(R) = m_0 \dot{\mathbf{v}}, \quad (1)$$

where  $\mathbf{F}_{\text{ext}}$  is the applied four-force,  $\mathbf{v}$  is the four-velocity of the shell, the dot signifies  $d/d\tau$  (where  $\tau$  is proper time), and  $q$ ,  $R$ , and  $m_0$  are the charge, proper radius, and bare mass of the shell ( $m_0$  includes a contribution from the internal stress as well as the energy density of the shell). It is customary to move the  $\dot{\mathbf{v}}/R$  term to the right-hand side, so as to obtain

$$\mathbf{F}_{\text{ext}} + \frac{2}{3}q^2(\ddot{\mathbf{v}} - \dot{\mathbf{v}}^2 \mathbf{v}) + O(R) = m \dot{\mathbf{v}}, \quad (2)$$

where  $m = m_0 + (2/3)q^2/R$  is called the *observed mass*. The equation is obtained from a power series expansion of the exact self-force. The neglected higher-order terms are small as long as the worldline does not curve significantly on the distance or time scale set by the size of the sphere  $R$ .

If we neglect the  $O(R)$  terms and set  $\mathbf{F}_{\text{ext}} = 0$  in Eq. (2), we have

$$\frac{2}{3}q^2(\ddot{\mathbf{v}} - \dot{\mathbf{v}}^2 \mathbf{v}) = m \dot{\mathbf{v}}. \quad (3)$$

This equation has the solution  $\dot{\mathbf{v}} = 0$  (as expected), and also another solution, whose behavior can be usefully captured by taking the low-velocity limit. In that limit the nonlinear term is negligible in the spatial part of the equation, and one finds that the 3-acceleration grows exponentially:

$$\dot{\mathbf{v}} \simeq \dot{\mathbf{v}}(0)e^{\Gamma\tau}, \quad (4)$$

where

$$\Gamma = \frac{3m}{2q^2}. \quad (5)$$

Equation (4) is accurate when  $v \ll \Gamma/a_0$ , where  $a_0$  is the proper acceleration. This is the famous runaway solution of the ALD equation; it defies physical sense because it suggests that the sphere (and the field) can acquire energy and momentum even when no external force acts. This solution has been much discussed because it was argued that the ALD equation should be exact in the point-particle limit  $R \rightarrow 0$ . In fact that limit does not make sense for finite  $q$  and  $m$ , as we argue more fully below. In any case, the solution respects the approximations assumed by the ALD equation only if  $\Gamma \ll 1/R$ , which implies

$$\frac{3mR}{2q^2} \ll 1, \quad (6)$$

hence

$$\frac{m_0 + m_{\text{ed}}}{m_{\text{ed}}} \ll 1, \quad (7)$$

where  $m_{\text{ed}} = (2/3)q^2/R$ . However, this condition cannot be satisfied for non-negative  $m_0$ . Thus, if the bare mass is greater than or equal to zero, the neglected higher-order terms are non-negligible for the worldline described by the runaway solution. This means the calculation has broken down, and the solution is merely an artifact of treating the exact equation of motion incorrectly.

Next, consider the exact relativistic equation of motion for a given small charged entity. The spatial part takes the form

$$\mathbf{f}_{\text{ext}} + \mathbf{f}_{\text{self}} + \mathbf{f}_{\text{P}} = \frac{d}{dt}(\gamma m_0 \mathbf{v}), \quad (8)$$

where  $m_0$  is the “completely bare” mass (i.e., before internal stresses are accounted for),  $\mathbf{f}_{\text{self}}$  is the electromagnetic self-force, and  $\mathbf{f}_{\text{P}}$  is the self-force owing to internal stresses in the object (Poincaré stresses). For clarity, we emphasize that  $\mathbf{f}_{\text{self}}$  includes all contributions to the force owing to the electromagnetic field sourced by the object in question. The object should not be considered to be point-like when calculating this force, and the inertial term in  $\mathbf{f}_{\text{self}}$  must not be neglected.

First let us consider rectilinear motion, in order to understand what gives rise to runaway or self-accelerating solutions in this case. For the case of rectilinear motion, by bringing the Poincaré term to the right hand side, one can always write

$$\mathbf{f}_{\text{ext}} + \mathbf{f}_{\text{self}} = \frac{d}{dt}(\gamma m_0 \mathbf{v}) \quad (9)$$

by defining the “bare mass”  $m_0$  appropriately. Note that  $m_0$  may be time-dependent.

In all cases that have ever been calculated,  $\mathbf{f}_{\text{self}}$  is opposed to  $\dot{\mathbf{v}}$  in the instantaneous rest frame. This being so, it is easy to see that if one allows  $m_0 \leq 0$  then one can find a solution with  $\dot{\mathbf{v}} \neq 0$  when  $\mathbf{f}_{\text{ext}} = 0$ , or, more generally, runaway solutions for a variety of forms of  $\mathbf{f}_{\text{ext}}$ . The source of this unphysical prediction is not some problem with classical electromagnetism, as many authors have suggested. It is simply an error arising from an incorrect assumption that a physical system can have negative inertial mass.

Caldirola’s equation of motion is an approximation to the exact equation of motion that includes more terms than the ALD equation and therefore may be expected to give more accurate predictions.<sup>3,4,8,10,24</sup> Caldirola’s equation of motion has no runaway solutions when  $m_0 > 0$ . This is well established, and in view of what we have just said about the exact equation of motion, it is wholly unsurprising. However, the condition continues to be regarded as surprising because it is often stated as a condition on the size rather than the mass of the body. For example, when we treat a spherical shell of charge  $q$ , the constraint can be written  $R > (2/3)q^2/mc^2$ , where  $m$  is observed mass, and Griffiths *et al.*<sup>11</sup> comment “it does seem peculiar that classical electrodynamics should harbor such a counterintuitive constraint on its validity.” We argue that, on the contrary, the constraint is not counterintuitive and its ultimate cause has nothing to do with electrodynamics *per se*.

## A. An illustrative example

Figure 1 shows a simple physical system that we shall use to illustrate the use and abuse of inertial mass in classical physics. The system consists of two small bodies of rest



Fig. 1. Illustrative system consisting of two masses separated by a fluid-filled tube.

masses  $m_1$  and  $m_2$ , separated by a tube containing an ideal fluid of pressure  $p$ . The tube and the fluid have negligible energy density. The whole system is uncharged; there are no electromagnetic fields in the problem. We suppose this system to be undergoing motion at constant proper acceleration (“hyperbolic motion”) in the direction parallel to the tube. We assume the motion is rigid, that is, it maintains the proper dimensions of the system. In the instantaneous rest frame, the equations of motion for the two bodies are

$$\begin{aligned} m_1 a_1 &= f_1 - A p_1, \\ m_2 a_2 &= f_2 + A p_2, \end{aligned} \quad (10)$$

where  $f_{1,2}$  are the external forces,  $p_{1,2}$  are the values of the pressure at the two ends of the tube, and  $A$  is the cross-sectional area of the latter.

To find the pressures, we solve the relativistic Navier-Stokes Equation (see Ref. 33), which gives

$$\frac{p_1}{p_2} = \frac{a_1}{a_2}. \quad (11)$$

Substituting this into Eqs. (10), we find that for  $f_1 = f_2 = 0$  (no external force) the system can nevertheless have  $a_1 > 0$  and  $a_2 > 0$  (self-acceleration) if and only if

$$m_1 = -m_2. \quad (12)$$

Thus, here we have an example of a classical system showing self-acceleration in the absence of any electromagnetic fields—if we admit the possibility of negative rest mass for part of the system. Note that the system as a whole does not have zero mass when  $m_1 = -m_2$ , because the pressure contributes an inertial mass of  $pV/c^2$ , where  $V$  is the volume of fluid in the tube. Thus, the system might appear to a naive examination to be “possible” when  $p > 0$ . when in fact it is not possible.

## B. Discussion: A lower bound on mass

Returning now to the case of a charged sphere, we are arguing that  $m_0$  must be positive because this is simply one of the assumptions of classical physics: inertia always opposes acceleration. However, in the case of a system permanently in possession of its electric charge, the “bare” system with its bare mass is just a convenient fiction introduced for the purposes of writing down an equation of motion. One might want to suggest that for such systems the observed mass could in principle take on any positive value. This is exactly what has been done in many discussions of runaway solutions to the equation of motion. However, those runaway solutions are themselves “giving the game away”: they are themselves a signal that an unphysical assumption has been smuggled in. The concept of bare mass is a helpful way to clarify what that assumption is.

Let  $m_{\text{ed}}$  be the contribution of the field to the observed mass of the system “charged object plus its own field.” (In

the case of a spherical shell,  $m_{\text{ed}} = (2/3)q^2/Rc^2$ .) The assumption  $m < m_{\text{ed}}$  amounts to assuming that the system in question is equivalent to one in which an electromagnetic field surrounds and interacts with an entity whose momentum opposes its velocity and whose kinetic energy is negative (because the mass of the entity is  $m - m_{\text{ed}}$ ). But no such entity exists or could exist. One should not treat really existing charged particles such as electrons as if they were equivalent to systems with unphysical internal properties such as negative inertial mass.

One might want to suggest that, in the point-particle limit, somehow the electromagnetic energy and the binding energy could result in  $m < m_{\text{ed}}$ . It might be proposed, for example, that  $m < m_{\text{ed}}$  could in some situations account correctly for the inertial effects when they are averaged over a small region containing a charged body that is stabilized by its own Poincaré stresses. However, this is not possible, because for each shell of charge the electromagnetic part of the energy resides in the electromagnetic field *outside* the shell, so the only way to get  $m < m_{\text{ed}}$  through effects inside a shell is to have a net *negative* rest energy inside the shell and this is not allowed. Another strategy to retain the concept of point-like charges would be to suggest that there is some further physical effect around the particle that lowers the total energy density there. This could be valid (it is indeed exactly what happens, according to quantum field theory), but one would then have to calculate the influence of this further effect on the self-force.

Another way to modify the vacuum outside the charged body is to allow for its gravitational field; this is considered in the next section.

Of course, all physical systems are “impossible” as far as classical mechanics is concerned, since they are actually described by quantum theory. However, the classical limit of quantum theory should be expected to consist in physically reasonable behavior and should be treated accordingly.<sup>34</sup> We simply have to accept that a point-like particle with finite charge and finite observed mass is a strict impossibility in classical physics [with the exception of a special case of charge-mass ratio given in Eq. (14)]; it is mathematically dubious and physically impossible—and others agree.<sup>13,16,17</sup> When we call an electron a “charged particle” we should keep in mind that that terminology can be misleading and in fact is misleading in the case of motion where self-force is significant. The excitations of the Dirac field do not behave like charged point-like classical particles, not even in the classical limit, and it is only a muddle in understanding that limit that led people to mistakenly think that they might. Electrons do behave, to good approximation, like charged rigid spheres of finite radius, as long as there is no significant structure in the applied fields and dynamics on the length scale (or associated time scale) of the sphere’s size. When those conditions are satisfied, there is no need to be specific about the size, as long as it is above the critical value, because it merely affects the way the observed mass is “parceled out” between two contributions ( $m_0$  and  $m_{\text{ed}}$ ) that are themselves unobservable. The latter are “hidden variables” of a classical model of a charged wave-packet; they have served their purpose as soon as they have revealed the critical length scale. (The critical length scale is the length scale where  $m = m_{\text{ed}}$ .)

The rigid spherical shell is one allowable approximate model of an electron, but of course it is not the only option. One could instead modify the Maxwell equations themselves.<sup>35,36</sup> One physically motivated option is to treat the

electron as a point particle but modify the treatment of the vacuum, by allowing that it can be polarized.<sup>36</sup>

### C. Gravitational effects

In the point-particle limit the gravitational as well as the electromagnetic field diverges, so a thorough investigation of that limit must take account of both. Using a simple Newtonian model, the gravitational energy liberated by bringing together material of mass  $m$  from initially infinite separation to form a spherical shell of radius  $R$  is  $Gm^2/2R$ . Therefore, one expects the total self-energy, including electrostatic and gravitational contributions, to be

$$\frac{1}{2R} \left( \frac{Q^2}{4\pi\epsilon_0} - Gm^2 \right), \quad (13)$$

where  $Q$  is the charge and for the purpose of this equation we have adopted SI units (in Gaussian units the first term in the bracket is  $q^2$ ). This self-energy vanishes when

$$\frac{Q}{m} = \sqrt{4\pi\epsilon_0 G} \simeq 8.618 \times 10^{-11} \text{ C kg}^{-1}, \quad (14)$$

so for this specific value of  $Q/m$  one can take the point limit  $R \rightarrow 0$  without meeting a divergence. However, this rescues the theory only for a particle having this special value of  $Q/m$ , so the point limit remains highly questionable in general; it certainly cannot be used as a model for electrons, whose charge/mass ratio is  $2 \times 10^{21}$  times larger than the special value.

Of course, the above can only be regarded as an order-of-magnitude estimate, but a rigorous treatment by Arnowitt *et al.*<sup>21,22</sup> comes to the conclusion that a stable and well-behaved solution of the field equations of general relativity is possible for a charged shell in the point limit, with the ratio of charge to observed mass given by Eq. (14).

## III. OBTAINING THE ELL/FO EQUATION

We now turn to the issue of obtaining an equation of motion for a small charged object.

Although the point charge (with finite charge but no extension) is a useful tool for pedagogic purposes when students are first introduced to electromagnetism, it must be dropped if we want to formulate a mathematically consistent and physically sensible treatment of the subject. Instead one can treat continuous distributions of charge with no impossible properties such as divergent total energy. Point particles can still be treated as long as they have vanishing charge and mass, as we illustrate below.

The idea is to obtain a general equation of motion for a small charged body that will be approximate, but whose approximation is satisfactory; that is, it predicts, in an appropriate limit, motion close to the exact solution. In particular, the equation of motion does not predict runaway behavior in the absence of an applied force, and it is second order (in derivatives of position with respect to time) so it respects causality. To achieve this, we shall keep the degree of approximation in view, and thus arrive at a physically sensible equation that can be written in terms of observable quantities such as observed mass, without requiring information about the size of the object (except the reassurance that it is small enough for the approximations to be valid). Such an

equation can then be used to treat the motion of entities such as electrons to good approximation.

There is more than one possible strategy to achieve our end. Here we display a strategy inspired by Gralla *et al.* but involving only very simple ideas.

First, we write down the equation of motion of a rigid spherical shell of charge  $q$  and bare mass  $m_0$ :

$$\mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{self}} = m_0 \dot{\mathbf{v}}, \quad (15)$$

where  $\mathbf{F}_{\text{self}}$  is given by the ALD Eq. (1):

$$\mathbf{F}_{\text{self}} = \frac{2}{3} q^2 \left( -\frac{\dot{\mathbf{v}}}{R} + \ddot{\mathbf{v}} - \dot{\mathbf{v}}^2 \mathbf{v} \right) + O(q^2 R). \quad (16)$$

If one wants higher accuracy, one can expand the self-force to higher order.<sup>1,37</sup> In any case, as long as the equation of motion is itself correct only up to some order in  $R$ , there is no need to find exact solutions. For whatever order in  $R$  one has obtained an expression for  $\mathbf{F}_{\text{self}}$ , it suffices to find solutions for  $\mathbf{v}(\tau)$  that are accurate to that order.

To this end, we will obtain the solution as a power series in  $R$ , but taking care to express the limit  $R \rightarrow 0$  in a mathematically and physically sensible manner. This can be done by treating the shell as an example of an object whose radius might take smaller values, with charge and bare mass scaling with  $R$  in such a way as to give sensible predictions. For example, both charge and mass should vanish in the limit  $R \rightarrow 0$ . Therefore, we propose

$$q = \sigma R^n, \quad m_0 = \rho R^{n'}, \quad (17)$$

where  $\sigma$ ,  $\rho$ ,  $n$ , and  $n'$  are constants with  $\rho, n, n' > 0$ . We also assume that  $\mathbf{F}_{\text{ext}}$  varies with  $R$  in a reasonable way. To be precise, we define the 4-force per unit charge

$$\tilde{\mathbf{F}}_{\text{ext}} \equiv \mathbf{F}_{\text{ext}}/q, \quad (18)$$

and we assume that  $\tilde{\mathbf{F}}_{\text{ext}}$  can be expressed by a Taylor expansion about  $R = 0$ :

$$\tilde{\mathbf{F}}_{\text{ext}}(R) = \sum_{k=0}^{\infty} \frac{\tilde{\mathbf{F}}_{\text{ext}}^{(k)}(0)}{k!} R^k. \quad (19)$$

The equation of motion is then

$$\begin{aligned} \sigma R^n \tilde{\mathbf{F}}_{\text{ext}}(0) + \sigma \sum_{k=1}^{\infty} \frac{\tilde{\mathbf{F}}_{\text{ext}}^{(k)}(0)}{k!} R^{n+k} + \frac{2}{3} \sigma^2 [-\dot{\mathbf{v}} R^{2n-1} \\ + (\ddot{\mathbf{v}} - \dot{\mathbf{v}}^2 \mathbf{v}) R^{2n} + O(R^{2n+1})] = \rho R^{n'} \dot{\mathbf{v}}. \end{aligned} \quad (20)$$

We are at liberty to choose the powers  $n$  and  $n'$  freely, as long as no unphysical or mathematically dubious case is implied. Therefore, we make the natural assumption  $n = n'$ —it means the shell is an example of a class of shells whose charge-to-bare-mass ratio is constant as the radius varies. (The assumption  $n > n'$  is also viable; it merely complicates the analysis.) Equation (20) has a well-behaved limit as  $R \rightarrow 0$  as long as  $2n - 1 \geq 0$ , so we assume

$$n \geq 1/2. \quad (21)$$

We can now find a solution curve by writing the Taylor series

$$\mathbf{v}(R, \tau) = \sum_{k=0}^{\infty} \frac{\mathbf{v}^{(k)}(0, \tau)}{k!} R^k. \quad (22)$$

We substitute this into Eq. (20) and, in a first approximation, keep only the lowest order terms to get

$$\sigma R^n \tilde{\mathbf{F}}_{\text{ext}}(0) = \dot{\mathbf{v}}_0 \left[ \rho R^n + \frac{2\sigma^2}{3} R^{2n-1} \right], \quad (23)$$

where  $\mathbf{v}_0 = \mathbf{v}(0, \tau)$ . Note that this equation is accurate to order  $R^\eta$  where  $\eta = \min(n, 2n - 1)$ ; the fact that it contains some higher-order terms when  $n \neq 1$  does not change this. An interesting case is  $n = 1$ , which makes all these terms of the same order and leads to the simplest analysis. However we don't require that  $n = 1$ , only that  $n \geq 1/2$ . With the help of Eq. (17), Eq. (23) can be written as

$$q \tilde{\mathbf{F}}_{\text{ext}}(0) = m \dot{\mathbf{v}}_0, \quad (24)$$

where

$$m = m_0 + \frac{2q^2}{3R} = m_0 + m_{\text{ed}}. \quad (25)$$

Equation (24) is a well-behaved equation of motion; it has precisely the form of the Lorentz force equation with no self-force, but with the electrodynamics contributing to the observed mass. Equation (25) warns us that to model a ‘‘particle’’ of given charge and observed mass, we shall need to assume  $R \geq 2q^2/3m$ , and therefore the predictions cannot be expected to be accurate if  $\mathbf{F}_{\text{ext}}$  or the worldline varies significantly on this distance or time scale. In particular, we require

$$\ddot{\mathbf{F}}_{\text{ext}} \ll \dot{\mathbf{F}}_{\text{ext}} m / q^2. \quad (26)$$

The next set of terms in the power series (22) is obtained by substituting (22) into the equation of motion (20), retaining now all terms up to  $O(R^{2n})$ :

$$\mathbf{F}_{\text{ext}} + \frac{2}{3} q^2 (\ddot{\mathbf{v}}_0 - \dot{\mathbf{v}}_0^2 \mathbf{v}_0) = m \dot{\mathbf{v}} + O(R^{2n+1}). \quad (27)$$

By writing the full  $\mathbf{F}_{\text{ext}}$  on the left-hand side and the full  $\dot{\mathbf{v}}$  on the right-hand side, we have also retained some terms of order higher than  $R^{2n}$  but this does not change the fact that this equation is valid to order  $R^{2n}$ . Note that, in common with Eq. (24), the equation can be written in terms of  $q$ ,  $m$ , and  $\mathbf{F}_{\text{ext}}$  without explicitly mentioning  $R$ . The result is an equation of motion of the ALD form, but in which the self-force terms on the left-hand side are given by using the lowest-order approximation to the worldline. In short, it is a ‘‘reduced order’’ ALD equation.

Because, in this method of solution, we already know  $\mathbf{v}_0$  by the time we attempt to solve Eq. (27), we may as well use the power series (22) on the right-hand side and thus simplify the equation a little, obtaining

$$\delta \mathbf{F}_{\text{ext}} + \frac{2}{3} q^2 (\ddot{\delta \mathbf{v}}_0 - \dot{\delta \mathbf{v}}_0^2 \mathbf{v}_0) = m \delta \dot{\mathbf{v}}, \quad (28)$$

where we have written  $\delta \mathbf{v} \equiv \mathbf{v} - \mathbf{v}_0$  and  $\delta \mathbf{F}_{\text{ext}} \equiv \mathbf{F}_{\text{ext}}(R) - \mathbf{F}_{\text{ext}}(0)$ . This equation gives the perturbation in the worldline owing to the self-force and the  $R$ -dependence of  $\mathbf{F}_{\text{ext}}$ . In this form, one must propose a value for  $R$  or use some other

strategy in order to obtain an expression for  $\delta F_{\text{ext}}$ , but this is in any case necessary in order to evaluate  $F_{\text{ext}}$  if  $F_{\text{ext}}$  depends on  $R$ .

Equation (27) is accurate to the order shown. However, it is convenient to adjust the equation, while still retaining the same order of accuracy, so as to make the left-hand side orthogonal to the four-velocity. Then when one drops those  $O(R^{2n+1})$  terms that are not implicitly included, one has a well-constructed four-vector equation, both sides of which are orthogonal to  $v$ . To this end, replace  $\dot{v}_0$  by  $\dot{F}_{\text{ext}}/m$  and one occurrence of  $\dot{v}_0$  by  $F_{\text{ext}}/m$ , and elsewhere replace  $v_0$  by  $v$ :

$$F_{\text{ext}} + \frac{2q^2}{3m} [\dot{F}_{\text{ext}} - (\dot{v} \cdot F_{\text{ext}})v] = m\dot{v}, \quad (29)$$

where we now take it as understood that the equation is accurate only to order  $O(R^{2n})$ . Taking the inner product with  $v$ , we obtain

$$F_{\text{ext}} \cdot v + \frac{2q^2}{3m} (\dot{F}_{\text{ext}} \cdot v + \dot{v} \cdot F_{\text{ext}}) = 0. \quad (30)$$

This equation is satisfied if the external force is pure, i.e., if  $F_{\text{ext}} \cdot v = 0$  at all times (this is true for electromagnetic forces). For the case  $F_{\text{ext}} = q\mathbb{F} \cdot v$ , where  $\mathbb{F}$  is the Faraday tensor of the applied electromagnetic field, Eq. (29) is Eliezer's equation (see appendix). This justifies the use of the Eliezer or Ford-O'Connell equation in calculating the motion of electrons and other charged particles. The Landau-Lifshitz equation is almost identical, and may also be used, because it is accurate to the same order of approximation (see the appendix for details).

The precision of the solution obtained by the above strategy is subject to the considerations outlined in Gralla *et al.*<sup>13</sup> As time goes on, the departure from the unperturbed worldline grows. Eventually the terms of any given order  $R^n$  are no longer small in comparison to those of lower order, and the approximations break down. One method to deal with this is to start a new iteration of the calculation after some not-too-great elapse of proper time, and thus work one's way along the worldline.

In passing from Eq. (27) to Eq. (29), the above treatment merely introduced replacements in a such a way as to retain the order of approximation while passing to a form where the higher-order terms might possibly be zero. One could also obtain the relativistic Landau-Lifshitz equation this way, so the present discussion does not offer any reason to prefer the equation of Eliezer or Ford and O'Connell to that of Landau and Lifshitz (though Ford and O'Connell have given reasons to prefer their equation<sup>30,38</sup>). One cannot expect either equation to be exact (within classical theory), because they do not account for details of the charge distribution that one expects to influence the equation of motion at higher order, but for a specific entity such as an electron there is a good chance that one equation is more accurate than the other. Experimental exploration of this difference is challenging, but may become possible using modern laser techniques. The question can also be explored by numerical (as well as analytical) treatments, since the whole framework of this problem remains fully within ordinary classical electromagnetism with no adjustments needed; the only thing that Maxwell's and Lorentz's equations cannot supply is a framework for calculating material properties such as

internal stresses (Poincaré stresses), which must be supplied by assuming an equation of state.

The ELL/FO equation is valuable because it is more accurate and more straightforward to work with than the ALD equation. The accuracy is higher because it accounts better for the charge distribution. It is also convenient because there is less danger of making an unphysical parameter choice such as a too-small value of  $R$  at given  $q$  and  $m$ , and because it is second order (when regarded as a differential equation for the 4-position), which makes it respect causality. The value of the above derivation (and others that arrive at the same conclusion) is that it shows that the ELL/FO equation should not be regarded as an approximation to the ALD equation; rather, it is an alternative and better approximation to the exact equation of motion for a rigid spherical charge. This is so because it is accurate to the same order in  $R$  while avoiding the properties that cause ALD to exhibit pathological behavior. See Refs. 11 and 28 for a comparison of their predictions for some example cases, and the accuracy limit associated with the condition (26).

#### IV. CONCLUSION

To sum up, in this paper, we have presented two main ideas. First we discussed the absence of runaway behavior in the correct treatment of the motion of charged bodies. The inertial term in the electromagnetic self-force is often included implicitly rather than explicitly in the equation of motion by absorbing it into the mass of the particles under consideration. This can be correct, but only if one keeps in mind the need to avoid unphysical parameter values, such as negative total rest energy in some region of space. In all cases that have been calculated, and total self-force of an accelerated object *opposes* the acceleration. Therefore, there is no known physical object, having a positive bare mass, whose equation of motion has runaway solutions in the absence of an applied force, when that motion is treated by standard classical electromagnetism (Maxwell's equations and the Lorentz force equation).

The second main idea of this paper was to present an approach to handling the ALD equation, by considering it to be what it is—an approximate equation of motion of a rigid charged shell, not an exact equation for point particles. We have given a straightforward method to form the  $R \rightarrow 0$  limit to the equation of motion in a physically and mathematically sensible manner. This is similar to the approach taken by Gralla *et al.*, but considerably simpler. The result is a reduced-order ALD equation [(27) or (29)], previously recommended by several other authors.

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#### APPENDIX: LANDAU-LIFSHITZ EQUATION

Equation (52) of Eliezer<sup>26</sup> is, in our notation,

$$m\dot{v} = q\mathbb{F} \cdot v + \frac{2q^3}{3m} \left[ \frac{d}{dt} (\mathbb{F} \cdot v) - v(\mathbb{F} \cdot v) \cdot \dot{v} \right]. \quad (A1)$$

The Landau-Lifshitz equation, which is Eq. (12) of Gralla *et al.*,<sup>13</sup> is, in our notation,

$$m\dot{\mathbf{v}} = q\mathbb{F} \cdot \mathbf{v} + \frac{2q^3}{3m} [\mathbf{w} + \mathbf{v}(\mathbf{v} \cdot \mathbf{w})], \quad (\text{A2})$$

where

$$\mathbf{w} = \dot{\mathbb{F}} \cdot \mathbf{v} + \frac{q}{m} \mathbb{F} \cdot (\mathbb{F} \cdot \mathbf{v}). \quad (\text{A3})$$

These equations are very similar but not identical. Since  $\mathbb{F}$  is antisymmetric, we have  $\mathbf{v} \cdot \mathbb{F} \cdot \mathbf{v} = 0$  and therefore

$$\dot{\mathbf{v}} \cdot (\mathbb{F} \cdot \mathbf{v}) = -\mathbf{v} \cdot \frac{d}{d\tau} (\mathbb{F} \cdot \mathbf{v}). \quad (\text{A4})$$

Substituting this into Eq. (A1), Eliezer's equation can be written as

$$m\dot{\mathbf{v}} = q\mathbb{F} \cdot \mathbf{v} + \frac{2q^3}{3m} [\tilde{\mathbf{w}} + \mathbf{v}(\mathbf{v} \cdot \tilde{\mathbf{w}})], \quad (\text{A5})$$

where

$$\tilde{\mathbf{w}} = \frac{d}{d\tau} (\mathbb{F} \cdot \mathbf{v}) = \dot{\mathbb{F}} \cdot \mathbf{v} + \mathbb{F} \cdot \dot{\mathbf{v}}. \quad (\text{A6})$$

Comparing this with Eq. (A3), we see that, to convert the Eliezer equation into the Landau-Lifshitz equation one may replace the acceleration  $\dot{\mathbf{v}}$  in the expression for  $\tilde{\mathbf{w}}$  by its value ignoring self force, i.e.,  $(q/m)\mathbb{F} \cdot \mathbf{v}$ . This results in an equation that is valid to the same order of approximation, and easier to work with. As long as one does not make the mistake of assuming either equation is exact, either one may be used.

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