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Radiation of de-excited electrons at large times in a strong electromagnetic plane wave



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H I G H L I G H T S

- Late time asymptotics of the solutions to the Lorentz–Dirac equation are studied.
- General properties of the total radiation power of electrons are established.
- The total radiation power equals a half the rest energy divided by the proper-time.
- Spectral densities of radiation formed on the late time asymptotics are derived.
- Possible experimental verification of the results is proposed.

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The late time asymptotics of the physical solutions to the Lorentz–Dirac equation in the electromagnetic external fields of simple configurations – the constant homogeneous field, the linearly polarized plane wave (in particular, the constant uniform crossed field), and the circularly polarized plane wave – are found. The solutions to the Landau–Lifshitz equation for the external electromagnetic fields admitting a two-parametric symmetry group, which include as a particular case the above mentioned field configurations, are obtained. Some general properties of the total radiation power of a charged particle are established. In particular, for a circularly polarized wave and constant uniform crossed fields, the total radiation power in the asymptotic regime is independent of the charge and the external field strength, when expressed in terms of the proper-time, and equals a half the rest energy of a charged particle divided by its proper-time. The spectral densities of the radiation power formed on the late time asymptotics are derived for a charged particle moving in the external electromagnetic fields of the simple configurations pointed above. This provides a

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simple method to verify experimentally that the charged particle has reached the asymptotic regime.

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1. Introduction

The upcoming experimental facilities will allow us to observe experimentally a distinct manifestation of the radiation reaction effects in strong electromagnetic waves [1,2]. Although a clear-cut signal of the radiation friction was detected already in 1946 [3] from the decreasing of electron's orbit in the betatron, the theoretical and experimental studies of other effects stemming from the radiation reaction are of importance especially for the strong fields where such effects are tangible or even dominate under certain circumstances. One of the consequences of the radiation reaction of charged particles in strong fields is their rapid de-excitation to the state with a minimum radiation. For a constant homogeneous external magnetic field this state is well known (see, e.g., [4]) and corresponds to a motion of charged particles along the magnetic field lines. In this paper, we continue the investigation of late time asymptotics of the evolution of charged particles started in [5] (see also [6–8]). At the large proper-times, a charged particle is de-excited and passes to the asymptotic regime where it emits soft photons. We are about to describe these asymptotic regimes for the electromagnetic fields of simple configurations – constant homogeneous fields, crossed fields, and a plane wave field – and to obtain the spectral density of the radiation power for the latter two cases.

As long as in the asymptotic regime the electrons are de-excited and the photons they radiate are chiefly soft (with the energies much lesser than the electron rest energy), the classical theory of radiation should work well [9–11]. The main instrument that we shall employ to investigate the asymptotics of the evolution of charged particles is the Lorentz–Dirac (LD) equation [12,13]. In spite of the fact that this equation is ill-famed, we shall show as a byproduct of our study that it can be used to obtain solid predictions about the behaviour of charged particles provided its physical solutions are only taken into account. The notion of a physical solution to the LD equation was introduced in [14] and later was elaborated by many authors (see, e.g., [5,15–23]). The interested reader may consult these references for details, especially Ref. [5], since we shall heavily rely on the results of [5]. Roughly, the physical solution to the LD equation is such a solution which is an analytic function of the coupling constant (the particle charge) near its zero value. This condition completely rules out runaway solutions to the LD equation. Using the physical solutions to the LD equation, we shall find the asymptotic behaviour of charged particles in the above mentioned field configurations and then, employing these asymptotics, shall obtain the spectral density of radiation.

Apart from the main goal outlined above, we shall also study some general properties of the motion of charged particles in these field configurations with the radiation reaction taken into account. In particular, we shall prove that for such external electromagnetic fields the radiation reaction force tends to diminish the total radiation power such that a charged particle strives for the trajectory with a minimum radiation. It is interesting to note that this property complies with the general principle of the least entropy production for non-equilibrium systems [24,25]. In addition, for a constant uniform external electromagnetic field the total radiation power turns out to be a monotonically decreasing function of time. Whereas for a plane electromagnetic wave of circular polarization and for a constant uniform crossed field, the total radiation power expressed in terms of the proper-time is independent of the charge and the external field strength at large proper-times. The only property of a charged particle, which determines this asymptotics, is its mass. Of course, in order to reach this asymptotic regime, the particle must be electrically charged, and the time needed to pass to the asymptotics depends on the charge and the external field strength. Bearing in mind that the late time asymptotics effectively describes (in a certain approximation) the result of the action of the S -matrix on a localized electron wave packet, we see that such an asymptotics is essentially nonperturbative. The properties of the spectral density of radiation formed of the asymptotics of the physical solutions to the LD equation are also rather curious. It turns out that for constant homogeneous crossed fields the maximum intensity of radiation at a given photon energy does not fall on the orbit plane as one

may naively expect for ultrarelativistic charged particles, but is directed at a certain angle to this plane. The explicit expression for this angle will be obtained. As for the radiation of a charged particle in a plane electromagnetic wave of constant amplitude, the intensity of this radiation formed on the late time asymptotics represents a system of rings of maxima and minima alternating each other when projected to the plane orthogonal to the direction of propagation of the wave. The arrangement of these rings depends on the photon energy and so the radiation pattern looks like a circular rainbow. The features of this pattern will be completely described.

Note that the numerical simulations of the radiation produced by a charged particle with the radiation reaction taken into account were performed recently, for example, in [2,26,27] (for the early attempts for the case of a constant homogeneous magnetic field see, e.g., [28]). In these papers, the so-called Landau–Lifshitz (LL) equation [29] was used to describe the dynamics of a charged particle. Since these studies were mostly numerical, the late time asymptotics were not investigated. Notice that for an intense electromagnetic field achievable at the present moment [1,2] the electrons go to the universal asymptotic regime very quickly (see for the estimates Section 5.1.1). Even one period of the electromagnetic wave is sufficient for this.

In the present paper, we shall give an analytical description of the motion and radiation of charged particles at the large proper-times using the LD equation. This equation is the exact classical effective equation of motion for electrically charged particles and it follows from the energy–momentum conservation law (see, e.g., [30]) as opposed to the LL equation violating this conservation law in general. From quantum electrodynamics viewpoint, the LD equation effectively describes the motion of a localized wave-packet of scalar charged particles in an external slowly varying (on the wave-packet scale) electromagnetic field, when the electron–positron pair production is negligible and the radiated photons are sufficiently soft. In a recent study [31], the nonrelativistic integro-differential equation [32] with a special electron form factor was employed to remedy the above mentioned deficiency of the LL equation. That is, we can secure the energy–momentum conservation law for the LL equation by introducing the electron form-factor of a special form and, consequently, transforming this differential equation to the integro-differential one. However, this form-factor is unknown in the relativistic case. Remark that the LD equation restricted to the subset of physical solutions also admits numerical simulations as the LL equation (see [15,20] and for a recent one [33]), although they are more time-consuming. The fact that the LD equation represents a correct energy–momentum balance between a charged particle and an electromagnetic field is especially important for the late time asymptotics when the approximation errors (being made in passing from the LD equation to the LL equation) are accumulated and may lead to a different result than the exact solution (see, for instance, the asymptotics of a charged particle motion in a constant homogeneous magnetic field given in [15] and also Eq. (18)).

The quantum derivation of the LD equation and the problem of infrared divergences were analysed in [34] for a plane electromagnetic wave background. Note also that, as was shown in [5], one can think of the exact physical solutions to the LD equation as the Borel summed perturbative solutions in λ (see Eq. (3) for notation), the first correction in λ being described by the LL equation. The explicit examples of the Borel summation technique applied to the perturbative solutions of the LD equation were given in [35].

The paper is organized as follows. In Section 2, the notation and general formulas regarding the motion of charged particles in classical electrodynamics are given. Then, in Section 3, we shall investigate the asymptotics of the physical solutions to the LD equation for a charged particle moving in constant homogeneous, crossed, or plane wave electromagnetic fields. To provide a comprehensive analysis of the asymptotic behaviour of the evolution, we shall also consider there the solutions to the LL equation for the same field configurations. It is known (see [5–8,36,37]) that the LL equation is exactly solvable for such external electromagnetic fields. This will give us a simple means to find the time needed for a charged particle to reach the asymptotic regime. In Section 3.1, we shall obtain the late time asymptotics for the motion of a charged particle in a constant homogeneous external electromagnetic field. In Section 3.2, the analogous analysis will be given for the field of a plane electromagnetic wave. Besides, in these two sections, we shall find the solutions to the LL equation for the electromagnetic fields admitting a two-parametric symmetry group and generalize slightly the results of [8,36]. Section 4 is devoted to the properties of the total radiation power of a charged particle.

Then, in Section 5.1, we shall thoroughly analyse the spectral density of the radiation power formed on the asymptotics and the possibilities for its observation in an experiment. After all, in Section 5.2, we shall find the major contribution to the spectral density of radiation of charged particles in the fields of linearly and circularly polarized electromagnetic wave of constant amplitude.

2. Notation

We shall use the following conventions and notation. We choose the system of units such that $c = \hbar = 1$. Despite the fact that our treatment is purely classical and the Planck constant will not appear in the formulas, it is convenient to introduce it artificially to make easier a comparison with the units used to characterize the experimental installations. The action functional for the charged particle with a charge e and a mass m interacting with the electromagnetic field A_μ on the Minkowski background $\mathbb{R}^{1,3}$ with the metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ takes the form

$$S[x(\tau), A(x)] = -m \int d\tau \sqrt{\dot{x}^2} - e \int d\tau A_\mu \dot{x}^\mu - \frac{1}{16\pi} \int d^4x F_{\mu\nu} F^{\mu\nu}, \tag{1}$$

where $F_{\mu\nu} := \partial_{[\mu} A_{\nu]}$ is the strength tensor of the electromagnetic field,

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{bmatrix}, \tag{2}$$

and $x^\mu(\tau)$, $\mu = \overline{0, 3}$, is the worldline of a charged particle. In the natural parameterization $\dot{x}^2 = 1$, the Lorentz–Dirac (LD) equation [12,13] becomes

$$m\ddot{x}_\mu = eF_{\mu\nu}\dot{x}^\nu + \frac{2}{3}e^2(\ddot{x}_\mu + \dot{x}^2\dot{x}_\mu). \tag{3}$$

It is convenient to express all the lengths in terms of the Compton wavelength m^{-1} of the charged particle, that is we make a transform

$$x^\mu \rightarrow m^{-1}x^\mu, \quad \tau \rightarrow m^{-1}\tau. \tag{4}$$

Then the LD equation is rewritten as

$$\ddot{x}_\mu = \bar{F}_{\mu\nu}\dot{x}^\nu + \lambda(\ddot{x}_\mu + \dot{x}^2\dot{x}_\mu), \quad \bar{F}_{\mu\nu} := em^{-2}F_{\mu\nu}, \quad \lambda := 2e^2/3, \tag{5}$$

where x^μ , τ , $\bar{F}_{\mu\nu}$, and λ are dimensionless quantities. In making estimates, it is useful to have in view that for these dimensionless quantities the lengths are measured in the Compton wavelengths, the field strengths are measured in the units of the Schwinger (critical) field E_0 , and the energy unit is the rest energy of a particle. For an electron we have

$$l_c \approx 3.86 \times 10^{-11} \text{ cm}, \quad t_c \approx 1.29 \times 10^{-21} \text{ s}, \quad m \approx 5.11 \times 10^5 \text{ eV}, \tag{6}$$

$$E_0 = \frac{m^2}{|e|\hbar} \approx 4.41 \times 10^{13} \text{ G}, \quad \omega := \frac{E}{E_0}, \quad \lambda = \frac{2\alpha}{3} \approx \frac{2}{411},$$

where E is the external electromagnetic field strength. The up-to-date accelerator facilities are able to accelerate the electrons up to the energies of the order of 50 GeV. The intensities of the laser fields, which are accessible at the present moment [2,38], are of the order of 10^{22} W/cm^2 with the photon energies about 1 eV. These data correspond to

$$\gamma \approx 10^5, \quad \omega \approx 1.47 \times 10^{-4}, \quad \Omega \approx 1.96 \times 10^{-6}, \quad \frac{\omega}{\Omega} \approx 75.0, \tag{7}$$

$$\lambda\omega \approx 7.14 \times 10^{-7}, \quad \lambda\omega^2 \approx 1.05 \times 10^{-10},$$

where γ is the Lorentz factor of the electron and Ω is the photon energy in the units of the rest energy of an electron. Henceforth, we omit the bars over the dimensionless electromagnetic fields $\bar{F}_{\mu\nu}$ and all the electromagnetic fields below are $\bar{F}_{\mu\nu}$.

3. Asymptotics of physical solutions

In this section, we obtain the asymptotics of the physical solutions to the LD equation in the electromagnetic fields admitting a two-parametric symmetry group. Namely, we generalize the results of [5] to the case of an arbitrary constant homogeneous field and the field of an electromagnetic wave of both linear and circular polarizations. For these field configurations (their concrete form will be specified below) we shall find not only the asymptotics of the physical solutions to the LD equation, but integrate by quadratures the so-called Landau–Lifshitz (LL) equation [29], which approximately describes the radiation reaction. The latter property (the integrability of the LL equation) is a consequence of the invariance of the electromagnetic field strength tensor under the two-parametric symmetry group. The LD equation also possesses this symmetry group. However, it is not completely integrable for these fields since it is a differential equation of a higher order as against the LL equation.

3.1. Constant homogeneous field

Let us consider, at first, a charged particle moving in a constant homogeneous electromagnetic field

$$\begin{aligned}
 F^{\mu\nu} &= \omega_1 e_0^\mu e_1^\nu + \omega_2 e_2^\mu e_3^\nu, \\
 F_{\mu\nu}^2 &= \omega_1^2 (e_\mu^0 e_\nu^0 - e_\mu^1 e_\nu^1) + \omega_2^2 (e_\mu^2 e_\nu^2 + e_\mu^3 e_\nu^3), \quad (e_\alpha e_\beta) = \eta_{\alpha\beta},
 \end{aligned}
 \tag{8}$$

where $e_\alpha^\mu, \alpha = \overline{0, 3}$, is a tetrad constituted by the eigenvectors of the tensor $(F^2)^\mu_\nu$. The vectors $e_{0,1}^\mu$ correspond to the eigenvalue ω_1^2 of the tensor $(F^2)^\mu_\nu$, while the vectors $e_{2,3}^\mu$ are associated with the eigenvalue $-\omega_2^2$. The eigenvalues are expressed through the Poincaré-invariants of the electromagnetic field $I_1 = \mathbf{E}^2 - \mathbf{H}^2$ and $I_2 = 2(\mathbf{E}\mathbf{H})$ in the following way:

$$\omega_1^2 = (\sqrt{I_1^2 + I_2^2} + I_1)/2, \quad \omega_2^2 = (\sqrt{I_1^2 + I_2^2} - I_1)/2.
 \tag{9}$$

The Lorentz transformations leaving the strength tensor (8) of the electromagnetic field intact form an Abelian group $SO(1, 1) \times SO(2)$ with the matrix representation

$$\Lambda^{\mu\nu} = (e_\mu^0 e_\nu^0 + e_\mu^1 e_\nu^1) \cosh \psi + e_\mu^0 e_\nu^1 \sinh \psi + (e_\mu^2 e_\nu^2 + e_\mu^3 e_\nu^3) \cos \varphi + e_\mu^2 e_\nu^3 \sin \varphi,
 \tag{10}$$

where ψ and φ are the group parameters and the round brackets at a pair of indices mean a symmetrization without 1/2.

It is useful to rewrite the LD equation (5) in terms of the variables adjusted to the action of the symmetry group

$$v_0 = \sqrt{u_e} \cosh \psi, \quad v_1 = \sqrt{u_e} \sinh \psi, \quad v_2 = \sqrt{u_h} \cos \varphi, \quad v_3 = \sqrt{u_h} \sin \varphi,
 \tag{11}$$

where $v_\alpha := (e_\alpha v)$ are the projections of the 4-momentum onto the vectors of the tetrad, $\varphi(\tau)$ and $\psi(\tau)$ define a curve on the symmetry group, and

$$u_e = v_0^2 - v_1^2, \quad u_h = v_2^2 + v_3^2, \quad u_e(\tau) - u_h(\tau) = 1.
 \tag{12}$$

Introducing the notation

$$\begin{aligned}
 \dot{u}_e &:= u_e \dot{b}_e, & \dot{u}_h &:= u_h \dot{b}_h, \\
 \bar{a}_e &:= \frac{\dot{v}_0^2 - \dot{v}_1^2}{u_e}, & \bar{a}_h &:= \frac{\dot{v}_2^2 + \dot{v}_3^2}{u_h},
 \end{aligned}
 \tag{13}$$

the LD equation can be written in the form of a system of the first order equations (for brevity, we do not denote $\dot{\psi}$ and $\dot{\varphi}$ as new variables)

$$\begin{aligned}
 \lambda \ddot{\psi} &= \dot{\psi} - \omega_1 - \lambda b_e \dot{\psi}, & \lambda \ddot{\varphi} &= \dot{\varphi} - \omega_2 - \lambda b_h \dot{\varphi}, \\
 \lambda \dot{b}_e &= b_e - \lambda b_e^2 - 2\lambda u_h (\bar{a}_e - \bar{a}_h), & \lambda \dot{b}_h &= b_h - \lambda b_h^2 - 2\lambda u_e (\bar{a}_e - \bar{a}_h), \\
 \lambda \dot{\bar{a}}_e &= 2(\bar{a}_e + \omega_1 \dot{\psi}) - \lambda b_e [2\bar{a}_e + (\bar{a}_e - \bar{a}_h) u_h], \\
 \lambda \dot{\bar{a}}_h &= 2(\bar{a}_h - \omega_2 \dot{\varphi}) - \lambda b_h [2\bar{a}_h + (\bar{a}_e - \bar{a}_h) u_e].
 \end{aligned}
 \tag{14}$$

The equations in the second line are dependent by virtue of the differential consequence of the mass-shell condition (12)

$$u_e b_e = u_h b_h. \tag{15}$$

Notice that Eqs. (14) are still valid at $\omega_{1,2} = \omega_{1,2}(\tau)$.

The system of Eqs. (13) and (14) possesses a stationary physical solution (at $\omega_{1,2} = \text{const}$)

$$\begin{aligned} u_e = 1, \quad u_h = 0, \quad b_e = 0, \quad b_h &= \frac{1-g}{\lambda}, \\ \bar{a}_e = -\omega_1^2, \quad \bar{a}_h = -\omega_1^2 + \frac{g(g-1)}{2\lambda^2}, \\ \dot{\psi} = \omega_1, \quad \dot{\phi} = \frac{\omega_2}{g}, \end{aligned} \tag{16}$$

where we have introduced the notation

$$g := 2^{-1/2} \left(1 + 4\lambda^2 \omega_1^2 + \sqrt{(1 + 4\lambda^2 \omega_1^2)^2 + 16\lambda^2 \omega_2^2} \right)^{1/2}. \tag{17}$$

On substituting to (11), this solution corresponds to a hyperbolic motion along the electric field vector when $\omega_1 \neq 0$. In the case $\omega_1 = 0$, the particle moves according to this solution along a straight line in the spacetime. The last formula in (16) gives the limiting value of the rotation frequency of a charged particle revolving around the magnetic field vector in the system of coordinates associated with the tetrad e_a^μ , where $\mathbf{H} \parallel \mathbf{E}$. The frequency is timed with respect to particle's proper-time. Due to the radiation reaction, this frequency is less than the cyclotron frequency ω_2 .

It is not difficult to find the physical solution to the system of Eqs. (14) in a small neighbourhood of the point (16). It has the form

$$[\delta u_h, \delta b_h, \delta \bar{a}_e, \delta \bar{a}_h, \delta \dot{\psi}, \delta \dot{\phi}] = [u_h(0), \delta b_h(0), \delta \bar{a}_e(0), \delta \bar{a}_h(0), \delta \dot{\psi}(0), \delta \dot{\phi}(0)] e^{\lambda^{-1}(1-g)\tau}, \tag{18}$$

where all the preexponential factors are expressed in terms of $u_h(0)$ as

$$\begin{aligned} \delta b_h(0) &= \frac{1-g}{\lambda} \frac{3g-1+4\lambda^2\omega_1^2(1+g^{-1})}{5g-4-4\lambda^2\omega_1^2g^{-1}} \frac{2g-1}{3g-1} u_h(0), \\ \delta \bar{a}_e(0) &= 2\omega_1^2(1-g^{-1})u_h(0), \\ \delta \bar{a}_h(0) &= \frac{g-1}{8\lambda^2} [g-2-8\lambda^2\omega_1^2(g^{-1}-g^{-2})]u_h(0) \\ &\quad + [3g(g-1)-4\lambda^2\omega_1^2(3-g^{-1})] \frac{\delta b_h(0)}{4\lambda(2g-1)}, \\ \delta \dot{\psi}(0) &= \omega_1(g^{-1}-1)u_h(0), \\ \delta \dot{\phi}(0) &= \frac{\lambda\omega_2}{g} \frac{\delta b_h(0)}{2g-1}. \end{aligned} \tag{19}$$

Note that the expression $\lambda^{-1}(1-g)$ entering the exponent is regular at $\lambda \rightarrow 0$. The corrections to the “frequencies” $\delta \dot{\psi}$ and $\delta \dot{\phi}$ are of the opposite sign as compared with the main contributions (16) and tend exponentially to zero. The asymptotics we have found generalizes the known expression [5] for the electromagnetic field with $I_2 = 0$ (see [15] for the case of a constant homogeneous magnetic field).

Now we consider the LL equation [29],

$$\dot{\nu}_\mu = F_{\mu\nu} \nu^\nu + \lambda (\dot{F}_{\mu\nu} \nu^\nu + F_{\mu\nu} F^{\nu\rho} \nu_\rho - \nu^\lambda F_{\lambda\nu} F^{\nu\rho} \nu_\rho \nu_\mu), \tag{20}$$

approximating the LD equation, for the same field configuration (8). Though the LL equation looks cumbersome in comparison with the LD equation, the LL equation can be completely integrated for a constant homogeneous electromagnetic field [8]. The solutions to the LL equation allow us to describe

approximately the dynamics of a charged particle on the whole interval of time. That, in turn, allows us to find the estimates for the time needed for the particle to go to the universal regime (18) in terms of the initial data and the values of the external fields.

Making the substitution (11) and conducting all the calculations as for the LD equation above, we come to the system of equation of the form (14) for the LL equation

$$(\psi - \lambda\omega_1)' = \omega_1, \quad (\varphi - \lambda\omega_2)' = \omega_2, \quad \dot{u}_e = -2\lambda(\omega_1^2 + \omega_2^2)u_e u_h. \tag{21}$$

These equations also hold when $\omega_{1,2}$ are variable. For the constant $\omega_{1,2}$, the system (21) is easily integrated

$$u_e = \left[1 - \frac{u_h(0)}{u_e(0)} e^{-2\lambda(\omega_1^2 + \omega_2^2)\tau} \right]^{-1}, \quad u_h = \left[\frac{u_e(0)}{u_h(0)} e^{2\lambda(\omega_1^2 + \omega_2^2)\tau} - 1 \right]^{-1}, \tag{22}$$

$$\psi = \psi(0) + \omega_1\tau, \quad \varphi = \varphi(0) + \omega_2\tau.$$

Recall that the quantities u_e and u_h are related by the mass-shell condition (12). The momentum components evolve according to Eq. (11). The time needed for a charged particle to reach the asymptotic regime (the hyperbolic or the helical motion) can be estimated as

$$\tau \gg [2\lambda(\omega_1^2 + \omega_2^2)]^{-1} \left| \ln \frac{u_h(0)}{u_e(0)} \right|. \tag{23}$$

In the leading order in λ , the asymptotic regime following from the solutions (22) coincides with the asymptotics (16), (19) of the physical solutions to the LD equation.

3.1.1. Other solutions to the LL equation with the same symmetry group

When $\omega_{1,2}$ are the functions of τ , the system of Eqs. (21) can be integrated by quadratures too. This fact can be employed to generate the exact solutions to the LL equation. Besides, as it was mentioned in [5], if one of $\omega_{1,2}$ vanishes and the other one has a special form (see below) then the system (21) is also integrable by quadratures. Namely, let us assume $\omega_1 = 0$. Then $\psi = const$ and, making an appropriate Lorentz transform, one can put $\psi = 0$. If $\omega_2 = \omega_2(x_0)$ in this system of coordinates, i.e., a charged particle moves in a uniform magnetic field, which depends on time and is directed along the x^1 axis perpendicular to the trajectory, then the system of Eqs. (21) is reduced to

$$(\varphi - \lambda\omega_2)' = \omega_2 u_e^{-1/2}, \quad u_e' = -2\lambda\omega_2^2(u_e - 1)u_e^{1/2}, \quad \dot{x}_0 = u_e^{1/2}, \tag{24}$$

where the prime denotes the derivative with respect to x_0 . These equations are integrable by quadratures and

$$u_e^{1/2} = \frac{1 + u_e^{1/2}(0) - (1 - u_e^{1/2}(0)) \exp \left[-2\lambda \int_0^{x_0} dx \omega_2^2(x) \right]}{1 + u_e^{1/2}(0) + (1 - u_e^{1/2}(0)) \exp \left[-2\lambda \int_0^{x_0} dx \omega_2^2(x) \right]}, \tag{25}$$

where we have put $x_0(0) = 0$. Analogously, when $\omega_2 = 0$, we can set $\varphi = 0$ by an appropriate Lorentz transform. If $\omega_1 = \omega_1(x_2)$ in this frame, i.e., a charged particle moves in a constant electric field, which is directed along the x^1 axis and depends only on x^2 , then the system (21) can be cast into the form

$$(\psi - \lambda\omega_1)' = \omega_1 u_h^{-1/2}, \quad u_h' = -2\lambda\omega_1^2(1 + u_h)u_h^{1/2}, \quad \dot{x}_2 = u_h^{1/2}, \tag{26}$$

where the prime means the derivative with respect to x_2 . These equations are also integrable by quadratures and

$$u_h^{1/2} = \tan \left[\arctan u_h^{1/2}(0) - \lambda \int_0^{x_2} dx \omega_1^2(x) \right], \tag{27}$$

where it is assumed that $x_2(0) = 0$. However, we should note that the electromagnetic fields considered here are not the exact vacuum solutions to the Maxwell equations.

3.2. Plane electromagnetic wave

Another one field configuration, which is invariant with respect to a two-parametric symmetry group, is the plane electromagnetic wave field. Namely, the electromagnetic field strength tensor

$$F^{\mu\nu} = \omega(x_-) e_-^{\mu} \left[e_1^{\nu} \cos \varphi(x_-) + e_3^{\nu} \sin \varphi(x_-) \right], \quad F_{\mu\nu}^2 = \omega^2(x_-) e_-^{\mu} e_-^{\nu}, \quad (28)$$

where $\omega(x_-)$ and $\varphi(x_-)$ are the arbitrary functions, $x_- := (e_-x)$, and

$$(e_a e_b) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad a, b = \{-, 1, 3\}, \quad (29)$$

is invariant with respect to the transformations generated by the two commuting matrices belonging to the Lorentz group

$$\Lambda_1^{\mu\nu} = \eta^{\mu\nu} + r_1 e_1^{\mu} e_-^{\nu} + \frac{r_1^2}{2} e_-^{\mu} e_-^{\nu}, \quad \Lambda_3^{\mu\nu} = \eta^{\mu\nu} + r_3 e_3^{\mu} e_-^{\nu} + \frac{r_3^2}{2} e_-^{\mu} e_-^{\nu}, \quad (30)$$

where r_1 and r_3 are the group parameters. These matrices act on the vectors e_a^{μ} as follows

$$\begin{aligned} \Lambda_{1,3v}^{\mu} e_-^{\nu} &= e_-^{\mu}, & \Lambda_{1v}^{\mu} e_3^{\nu} &= e_3^{\mu}, & \Lambda_{1v}^{\mu} e_1^{\nu} &= e_1^{\mu} + r_1 e_-^{\mu}, \\ \Lambda_{3v}^{\mu} e_1^{\nu} &= e_1^{\mu}, & \Lambda_{3v}^{\mu} e_3^{\nu} &= e_3^{\mu} + r_3 e_-^{\mu}, \end{aligned} \quad (31)$$

whence the invariance of the tensor (28) under these transformations is evident. Note that the case of a linearly polarized wave corresponds to the choice $\varphi = 0$.

As in the case of a constant homogeneous electromagnetic field, it is useful to introduce the variables adjusted to the action of the symmetry group. Every vector of the Minkowski space can be expressed as a linear combination of the tetrad vectors

$$x^{\mu} = \frac{1}{2}(x_+ e_-^{\mu} + x_- e_+^{\mu}) - x_1 e_1^{\mu} - x_3 e_3^{\mu}, \quad x^2 = x_+ x_- - x_1^2 - x_3^2, \quad (32)$$

where e_+^{μ} is the isotropic vector orthogonal to e_1^{μ} and e_3^{μ} and such that $(e_+ e_-) = 2$. Hence, the action of the symmetry group (30) on the velocity 4-vector can be parameterized as

$$v_1 = v_- r_1, \quad v_3 = v_- r_3, \quad v_+ = v_-^{-1} + v_- (r_1^2 + r_3^2), \quad (33)$$

where the functions $r_1(\tau)$ and $r_3(\tau)$ determine a curve on the symmetry group. Using these variables, the LD equation is reduced to

$$\begin{aligned} \dot{v}_- &= \lambda \left\{ \ddot{v}_- - \left[\frac{\dot{v}_-^2}{v_-} + v_-^3 (\dot{r}_1^2 + \dot{r}_3^2) \right] \right\}, & \dot{v}_- &= - \int_{\tau}^{\infty} ds e^{(\tau-s)/\lambda} \left[\frac{\dot{v}_-^2}{v_-} + v_-^3 (\dot{r}_1^2 + \dot{r}_3^2) \right], \\ \dot{r}_1 &= \lambda \ddot{r}_1 + 2\lambda (\ln v_-) \dot{r}_1 + \omega \cos \varphi, & \dot{r}_1 v_-^2 &= \int_{\tau}^{\infty} \frac{ds}{\lambda} e^{(\tau-s)/\lambda} v_-^2 \omega \cos \varphi, \\ \dot{r}_3 &= \lambda \ddot{r}_3 + 2\lambda (\ln v_-) \dot{r}_3 + \omega \sin \varphi, & \dot{r}_3 v_-^2 &= \int_{\tau}^{\infty} \frac{ds}{\lambda} e^{(\tau-s)/\lambda} v_-^2 \omega \sin \varphi, \\ \dot{x}_- &= v_-, \end{aligned} \quad (34)$$

where all the subintegral functions are taken at the proper-time s . The integro-differential equations are obtained in the standard way (see, e.g., [15–17]) from their differential counterparts on the left restricting the LD equation to the subset of its physical solutions. Since

$$-\ddot{x}^2 = \frac{\dot{v}_-^2}{v_-^2} + v_-^2 (\dot{r}_1^2 + \dot{r}_3^2), \quad (35)$$

and $v_- > 0$, the integrand of the integro-differential equation in the first line of (34) is nonnegative, vanishes only for a rectilinear uniform motion, and tends to zero when $v_- \rightarrow 0$. Consequently,

v_- decreases until either the particle becomes moving along a straight line in the spacetime or $v_- \rightarrow 0$. A rectilinear uniform motion is not a solution to particle’s equations of motion in the electromagnetic wave field. Therefore, only the last case survives. The point $v_- = 0$ is an attractor for the physical solutions to the LD equation and so, for such field configurations, a charged particle will eventually move with a velocity close to the speed of light along the direction of propagation of the electromagnetic wave (in our case, along the x^2 axis in the system of coordinates associated with the tetrad).

In order to proceed, we perform the another one change of variables

$$\dot{r}_1 =: \rho \cos(\psi + \varphi), \quad \dot{r}_3 =: \rho \sin(\psi + \varphi), \quad b := (\ln v_-)'. \tag{36}$$

Then the LD equation becomes

$$\begin{aligned} \lambda \rho v_- \psi' &= \omega \sin \psi - \lambda \rho v_- \varphi', \\ \lambda v_- \rho' &= -\omega \cos \psi + (1 - 2\lambda b)\rho, \quad \rho v_-^2 = \int_{\tau}^{\infty} \frac{ds}{\lambda} e^{(\tau-s)/\lambda} v_-^2 \omega \cos \psi, \\ \lambda v_- b' &= b + \lambda \rho^2 v_-^2, \quad b = - \int_{\tau}^{\infty} \frac{ds}{\lambda} e^{(\tau-s)/\lambda} \rho^2 v_-^2, \\ v_-' &= b, \end{aligned} \tag{37}$$

where the prime denotes the derivative with respect to x_- . Recall that φ' is a given function of x_- . In the course of evolution $\omega(x_-)$ tends to the constant value ω_0 , the above system of the first order differential equations possesses a singular physical point

$$v_- = 0, \quad b = 0, \quad \psi = \pi n, \quad \rho = (-1)^n \omega_0, \tag{38}$$

where the integer number n is chosen such that $(-1)^n \omega_0 > 0$. The integro-differential equations (37) allow one to derive useful constraints on the parameters of the particle trajectory when the electromagnetic wave has a constant amplitude $\omega_0 > 0$. In this case, bearing in mind that v_- is a nonincreasing function, we obtain

$$\rho \leq v_-^{-2}(\tau) \int_{\tau}^{\infty} \frac{ds}{\lambda} e^{(\tau-s)/\lambda} v_-^2(s) \omega_0 \leq \omega_0 \int_{\tau}^{\infty} \frac{ds}{\lambda} e^{(\tau-s)/\lambda} \leq \omega_0, \quad 0 \geq b \geq -\omega_0^2 v_-^2. \tag{39}$$

Recollecting the definition of b , the last inequality can be integrated

$$v_-(0) \geq v_- \geq \frac{v_-(0)}{\sqrt{1 + 2\lambda \omega_0^2 v_-^2(0)\tau}}. \tag{40}$$

Later on we shall see that the second inequality in this formula turns into the equality for the solutions to the LL equation in the field of a circularly polarized plane electromagnetic wave of constant amplitude ω_0 (see Eq. (52)).

Now we find the asymptotics of the physical solutions to the LD equation for a particular case of a circularly polarized plane electromagnetic wave of the constant amplitude

$$\omega = \text{const} > 0, \quad \varphi = \Omega x_- + \varphi_0. \tag{41}$$

It is not difficult to find this asymptotics from the system of equations (37). At large times, the physical solution must tend to the physical singular point (38). Substituting the series developments of v_- , b , ρ , and ψ in terms of the inverse powers of x_- (it is clear from (40) that $x_- \rightarrow +\infty$ when $\tau \rightarrow +\infty$) with constant coefficients and the “initial data” (38), we come to

$$\begin{aligned} v_- &= \frac{1}{\lambda \omega^2 x_-} + \frac{k}{x_-^2} + O(x_-^{-3}), \\ \rho &= \omega - \left(1 + \frac{\Omega^2}{4\omega^2}\right) \frac{2}{\omega x_-^2} - \left(1 + \frac{\Omega^2}{4\omega^2}\right) \frac{4\lambda \omega k}{x_-^3} + O(x_-^{-4}), \\ \psi &= \frac{\Omega}{\omega^2 x_-} + \frac{\lambda \Omega k}{x_-^2} + O(x_-^{-3}). \end{aligned} \tag{42}$$

The asymptotics of the physical solutions (42) is parameterized by the only one arbitrary constant k , as it should be. Indeed, on integrating (36), the two additional arbitrary constants appear in the expressions for the 4-velocities. Integrating these 4-velocities, we obtain a set of solutions parameterized by the six arbitrary constants, that is (42) describes all the physical solutions in a vicinity of the singular point (38). In the asymptotic regime, the dependence of the solutions on the constant k disappears. Remark that the higher terms of the expansion (42) do not vanish at $k = 0$.

Consider the behaviour of a charged particle in the asymptotic regime with a more detail. For definiteness, we suppose that the particle strikes the electromagnetic wave at $x_0(0) = x_2(0) = 0$. Integrating the asymptotics (42), one can obtain in the leading order

$$\begin{aligned}
 x_- &= \left(\frac{2\tau}{\lambda\omega^2}\right)^{1/2} + O(\tau^{-1/2}), & v_- &= (2\lambda\omega^2\tau)^{-1/2} + O(\tau^{-3/2}), \\
 v_1 &= \frac{\omega}{\Omega} \sin \varphi + O(x_-^{-1}), & v_3 &= -\frac{\omega}{\Omega} \cos \varphi + O(x_-^{-1}), \\
 v_+ &= \lambda\omega^2 x_- \left(1 + \frac{\omega^2}{\Omega^2}\right) + O(1), \\
 x_0 &= \frac{\sqrt{2\lambda\omega^2}}{3} \left(1 + \frac{\omega^2}{\Omega^2}\right) \tau^{3/2} + O(\tau).
 \end{aligned}
 \tag{43}$$

The dependence of the trajectory on the initial velocity of the particle is lost in the limit of large proper-times. As we see, after the lapse of a certain time, the charged particle will move along the direction of propagation of the electromagnetic wave with the velocity close to the speed of light [8]. The transverse momentum components will rotate in a circle and will be cophased with the electric field vector of the electromagnetic wave. The radius of this circle equals ω/Ω . In the coordinate space, the particle moves along a helix with the axis parallel to the x^2 axis. The radius of the “circle”, which the particle sweeps up in the (x^1, x^3) plane, grows linearly with x_- . Inasmuch as the particle crosses the constant phase surfaces of the electromagnetic wave non-uniformly, the frequency of this rotation (the derivative of Ωx_-) declines as $\tau^{-1/2}$ with respect to the proper-time and as $x_0^{-2/3}$ with respect to the laboratory time.

For a linearly polarized wave of the constant amplitude ω_0 ,

$$\omega = \omega_0 \cos \psi, \quad \psi := \Omega x_- + \psi_0, \quad \varphi = 0,
 \tag{44}$$

the system of equations (34) is written as

$$\lambda(\rho v_-^2)' = v_-(\rho - \omega_0 \cos \psi), \quad \lambda v_- v_-'' = v_-^2 + \lambda \rho^2 v_-^2, \quad \dot{r}_3 = 0,
 \tag{45}$$

where $\rho \equiv \dot{r}_1$. We shall seek for a solution to these equations in the form of the series

$$\rho = \omega_0 \cos \psi + \rho_1/x_- + \rho_2/x_-^2 + \dots, \quad v_- = u_1/x_- + u_2/x_-^2 + \dots,
 \tag{46}$$

where ρ_i and u_i are the functions of x_- bounded with all their derivatives on the interval $x_- \in [0, +\infty)$. The leading order of the expansion (46) is determined by (45) at $\lambda = 0$. Substituting the series (46) into (45) and collecting the terms at the same power of x_- , we obtain for the first and second equations

$$\begin{aligned}
 u_1[\rho_1 + \lambda\omega_0\Omega u_1 \sin \psi - 2\lambda\omega_0 u_1' \cos \psi]/x_-^2 + \dots &= 0, \\
 u_1'/x_- + [u_2 - (1 + \lambda u_1'')u_1 + \lambda\omega_0^2 u_1^2 \cos^2 \psi]/x_-^2 + \dots &= 0,
 \end{aligned}
 \tag{47}$$

respectively. As long as the functions ρ_i and u_i are bounded with all their derivatives, the terms at the different powers of x_- should vanish independently. The first term of the second equation entails $u_1 = \text{const}$. The second term of the second equation gives us u_2 . Moreover, the boundedness of u_2 at $x_- \rightarrow +\infty$ can be provided if and only if $u_1 = 2(\lambda\omega_0^2)^{-1}$. Then we find ρ_1 from the first equation in

(47). If we continue this process, we arrive at

$$\begin{aligned} \rho &= \omega_0 \cos \psi - \frac{2\Omega \sin \psi}{\omega_0 x_-} - \frac{2k\Omega \omega_0^4 \sin \psi + 5\omega_0^2 \cos 3\psi + (8\Omega^2 + 11\omega_0^2) \cos \psi}{2\omega_0^3 x_-^2} \\ &\quad + O(x_-^{-3}), \\ v_- &= \frac{2}{\lambda \omega_0^2 x_-} + \left(k - \frac{\sin 2\psi}{\Omega \omega_0^2} \right) (\lambda x_-^2)^{-1} + O(x_-^{-3}), \end{aligned} \tag{48}$$

where k is an arbitrary constant. Thus we have found the asymptotics of all the physical solutions to Eqs. (45) at $x_- \rightarrow +\infty$. Assuming $x_0(0) = x_2(0) = 0$ and integrating the asymptotics above, we have

$$\begin{aligned} x_- &= 2 \left(\frac{\tau}{\lambda \omega_0^2} \right)^{1/2} + O(\tau^{-1/2}), \quad v_- = (\lambda \omega_0^2 \tau)^{-1/2} + O(\tau^{-3/2}), \\ v_1 &= \frac{\omega_0}{\Omega} \sin \psi + O(x_-^{-1}), \quad v_3 = 0, \quad v_+ = \frac{\lambda \omega_0^2 x_-}{2} \left(1 + \frac{\omega_0^2}{\Omega^2} \sin^2 \psi \right) + O(1), \\ x_0 &= \frac{\sqrt{\lambda \omega_0^2}}{3} \left(1 + \frac{\omega_0^2}{2\Omega^2} \right) \tau^{3/2} + O(\tau). \end{aligned} \tag{49}$$

The asymptotics of the 4-velocity differs from the analogous asymptotics (43) in the field of the circularly polarized wave only by the replacement $\omega_0^2 \rightarrow 2\omega^2$ and $\sin^2 \psi \rightarrow 1/2$ in the corresponding places. In other words, being averaged over the oscillation period of the wave, these asymptotics for the 4-velocities coincide.

Let us turn to the solution of the LL equation (20) for the electromagnetic field (28) with the arbitrary functions $\omega(x_-)$ and $\varphi(x_-)$. The substitution (33) reduces this equation to the system

$$\begin{aligned} \dot{v}_- &= -\lambda \omega^2 v_-^3, \quad (r_1 - \lambda \omega \cos \varphi)' = \omega \cos \varphi, \\ (r_3 - \lambda \omega \sin \varphi)' &= \omega \sin \varphi, \quad \dot{x}_- = v_-, \end{aligned} \tag{50}$$

which is integrable by quadratures [6–8,36,37] in an evident way. It is convenient to rewrite this system taking x_- as an evolutionary parameter. Then

$$\begin{aligned} v_- &= v_-(0) \left[1 + \lambda v_-(0) \int_0^{x_-} dx \omega^2(x) \right]^{-1}, \\ r_1 - \lambda \omega \cos \varphi &= r_1(0) - \lambda \omega(0) \cos \varphi(0) + v_-^{-1}(0) \\ &\quad \times \int_0^{x_-} dx \omega(x) \cos \varphi(x) \left[1 + \lambda v_-(0) \int_0^x dy \omega^2(y) \right], \\ r_3 - \lambda \omega \sin \varphi &= r_3(0) - \lambda \omega(0) \sin \varphi(0) + v_-^{-1}(0) \\ &\quad \times \int_0^{x_-} dx \omega(x) \sin \varphi(x) \left[1 + \lambda v_-(0) \int_0^x dy \omega^2(y) \right], \end{aligned} \tag{51}$$

where we have put, for definiteness, $x_-(0) = 0$. The first equation determines $x_-(\tau)$ by a quadrature.

In particular, for a circularly polarized electromagnetic wave of constant amplitude (41), we obtain

$$\begin{aligned} v_- &= \frac{v_-(0)}{\sqrt{1 + 2\lambda \omega^2 v_-^2(0) \tau}}, \quad x_- = \frac{\sqrt{1 + 2\lambda \omega^2 v_-^2(0) \tau} - 1}{\lambda \omega^2 v_-(0)}, \\ r_1 &= r_1(0) + \lambda \omega \left(1 + \frac{\omega^2}{\Omega^2} \right) (\cos \varphi - \cos \varphi_0) + \frac{\omega}{\Omega} \left[\lambda \omega^2 x_- \sin \varphi + \frac{\sin \varphi - \sin \varphi_0}{v_-(0)} \right], \\ r_3 &= r_3(0) + \lambda \omega \left(1 + \frac{\omega^2}{\Omega^2} \right) (\sin \varphi - \sin \varphi_0) - \frac{\omega}{\Omega} \left[\lambda \omega^2 x_- \cos \varphi + \frac{\cos \varphi - \cos \varphi_0}{v_-(0)} \right]. \end{aligned} \tag{52}$$

In the last two expressions, the first term in the square brackets dominates at large proper-times. Now the characteristic times needed for the charged particle to go to the asymptotic regime can be readily estimated. The projection v_- of the momentum onto the isotropic vector and the corresponding coordinate cease to depend on the initial data at the proper-times

$$\tau \gg [2\lambda\omega^2 v_-^2(0)]^{-1}. \tag{53}$$

Then the first term in the square brackets in the last two expressions of (52) prevails over all the other terms in these expressions provided that

$$\tau \gg \frac{\lambda\omega^2}{2\Omega^2}, \quad \tau \gg \frac{\lambda\Omega^2}{2\omega^2}, \quad \tau \gg \frac{\Omega^2}{2\lambda\omega^4} r_{1,3}^2(0). \tag{54}$$

When these inequalities are fulfilled, the asymptotics of the solutions to the LL equation coincides with the leading asymptotics of the physical solutions to the LD equation (43). Notice that the first inequality in (54) is equivalent to $\Omega x_- \gg 1$ and is satisfied with a good accuracy when the particle has crossed two “humps”, at least, of the electromagnetic wave.

The analogous calculations for a linearly polarized wave (44) give

$$\begin{aligned} v_- &= v_-(0) \left[1 + \frac{\lambda\omega_0^2 v_-(0)}{4\Omega} (2\Omega x_- + \sin 2\psi - \sin 2\psi_0) \right]^{-1}, \\ x_- \left(1 - \frac{\lambda\omega_0^2 v_-(0)}{4\Omega} \sin 2\psi_0 \right) + \frac{\lambda\omega_0^2 v_-(0)}{4\Omega^2} (\Omega^2 x_-^2 + \sin^2 \psi - \sin^2 \psi_0) &= v_-(0)\tau, \\ r_3 &= r_3(0), \\ r_1 &= r_1(0) + \lambda\omega_0 \left(1 + \frac{\omega_0^2}{2\Omega^2} \right) (\cos \psi - \cos \psi_0) + \left(\frac{\omega_0}{\Omega v_-(0)} - \frac{\lambda\omega_0^3}{4\Omega^2} \sin 2\psi_0 \right) \\ &\quad \times (\sin \psi - \sin \psi_0) + \frac{\lambda\omega_0^3}{2\Omega} x_- \sin \psi. \end{aligned} \tag{55}$$

The conditions that the charged particle reached the asymptotic regime are equivalent to (53) and (54). The corresponding asymptotic solution agrees with the asymptotics (49) of the physical solutions to the LD equation.

4. Radiation power

Let us analyse the behaviour of the radiation power of a charged particle moving along the trajectories derived in the previous section. At first, we obtain the total radiation power in the asymptotic regime for the charged particle evolving in the crossed electromagnetic fields [5]

$$v_1 \approx \text{sgn } \omega \left(\frac{\tau}{2\lambda} \right)^{1/2}, \quad v_- \approx (2\lambda\omega^2\tau)^{-1/2}, \quad v_+ \approx \left(\frac{\omega^2\tau^3}{2\lambda} \right)^{1/2}. \tag{56}$$

Under the term the crossed fields we always mean the electromagnetic fields of the form (28) with $\varphi = 0$ and $\omega = \text{const}$. Using the asymptotics above, we derive

$$\mathcal{P} = -\lambda\ddot{x}^2 \approx \lambda\omega^2 v_-^2 \approx \frac{1}{2\tau} \equiv \frac{mc^2}{2\tau} \approx \frac{1}{2} \left(\frac{\omega^2}{50\lambda x_0^2} \right)^{1/5}. \tag{57}$$

In the penultimate equality, we have restored the velocity of light and reverted to the dimensional quantities. Observe a curious property of the total radiation power. When it is expressed in terms of particle’s proper-time, it depends only on the mass of the particle in the asymptotic regime and is independent of the charge and the external field strength. Roughly, the value of this proper-time can be estimated from the exact solution to the LL equation (22), or (52). This solution implies that

τ is counted from the moment when the particle entered the region filled with the crossed electromagnetic field. As a function of the laboratory time x_0 , the total radiation power does not depend on the particle charge and is determined by the particle mass and the external field since $\omega \sim e$ and $\lambda \sim e^2$. The total radiation power evaluated by using the asymptotics (43) for a circularly polarized plane electromagnetic wave of constant amplitude turns out to be equal to

$$\mathcal{P} = -\lambda \ddot{x}^2 \approx \lambda \omega^2 v_-^2 \approx \frac{1}{2\tau} \approx \frac{(\lambda \omega^2)^{1/3}}{(6x_0)^{2/3}} \left(1 + \frac{\omega^2}{\Omega^2}\right)^{2/3}. \tag{58}$$

As in the case of the crossed fields, the total radiation power written in terms of the proper-time is independent of the particle charge and the external field strength in the asymptotic regime. Analogously, for a linearly polarized plane electromagnetic wave of constant amplitude, we have from the asymptotics (49),

$$\mathcal{P} = -\lambda \ddot{x}^2 \approx \lambda \omega_0^2 v_-^2 \cos^2 \psi \approx \frac{\cos^2 \psi}{\tau} \approx \frac{(\lambda \omega_0^2)^{1/3}}{(3x_0)^{2/3}} \left(1 + \frac{\omega_0^2}{2\Omega^2}\right)^{2/3} \cos^2 \psi. \tag{59}$$

On averaging over the oscillation period of the wave, the total radiation power of a charged particle moving in the linearly polarized wave passes into the corresponding expressions (57) or (58), when the expression for the power is written in terms of the proper-time.

Now we establish some general properties of the total radiation power for the motion in the electromagnetic fields of the form (8) and (28). It is well known [39] that the total radiation power in a constant homogeneous electromagnetic field is an integral of motion for the Lorentz equations

$$\ddot{x}_\mu = F_{\mu\nu} \dot{x}^\nu \Rightarrow \ddot{x}_\mu = \dot{F}_{\mu\nu} \dot{x}^\nu + F_{\mu\nu} \ddot{x}^\nu \Rightarrow \frac{1}{2} (\ddot{x}\ddot{x}) \cdot = \ddot{x}^\mu \dot{F}_{\mu\nu} \dot{x}^\nu = 0. \tag{60}$$

Also, using the representation (33), we obtain for the electromagnetic fields of the form (28)

$$\ddot{x}^\mu \dot{F}_{\mu\nu} \dot{x}^\nu = -v_-^2 [(\omega \cos \varphi) \dot{r}_1 + (\omega \sin \varphi) \dot{r}_3]. \tag{61}$$

Hence, taking into account the Lorentz equations (see (34) with $\lambda = 0$)

$$\dot{r}_1 = \omega \cos \varphi, \quad \dot{r}_3 = \omega \sin \varphi, \quad \dot{v}_- = 0, \tag{62}$$

we have

$$\ddot{x}^\mu \dot{F}_{\mu\nu} \dot{x}^\nu = -\frac{1}{2} v_-^2 (\omega^2) \cdot = -\frac{1}{2} (v_-^2 \omega^2) \cdot. \tag{63}$$

As we see, the quantity

$$\ddot{x}^2 + v_-^2 \omega^2 \tag{64}$$

is the integral of motion for the Lorentz equations with the electromagnetic fields of the form (28). However, as long as

$$\ddot{x}^2 = -\dot{x}^\mu F_{\mu\nu}^2 \dot{x}^\nu = -v_-^2 \omega^2 \tag{65}$$

for the Lorentz equations, this integral of motion is trivial. On the other hand, if ω is a constant on particle's trajectory, for example, for a circularly polarized plane electromagnetic wave of constant amplitude, then the total radiation power is the integral of motion of the Lorentz equations (recall that $v_- = const$). Comparing (65) with (57)–(59), remark an interesting property: according to the LD equation, the total radiation power of a charged particle in the asymptotic regime formally coincides with (65), i.e., at large proper-times, the contribution of the LD force to \ddot{x}^2 is negligible. This is the reason why the asymptotics of the solutions to the LL equation is the same as the asymptotics of the physical solutions to the LD equation for such field configurations.

When the radiation reaction is “turned on”, the total radiation power ceases to be an integral of motion for the fields of the form (8) and (28). For example, differentiating the LD equation (5) written for constant homogeneous external fields and convolving the result with \ddot{x}^μ , we arrive at

$$\frac{1}{2} \frac{d}{d\tau} \left[\dot{x}^2 - \lambda \frac{d}{d\tau} \ddot{x}^2 \right] = -\lambda F_{LD}^2, \quad F_{LD}^\mu := \ddot{x}^\mu + \dot{x}^2 \dot{x}^\mu. \tag{66}$$

Integrating both parts of this equation, we obtain

$$\ddot{x}^2(s) - \lambda \frac{d}{ds} \ddot{x}^2(s) = \ddot{x}^2(0) - \lambda \frac{d}{d\tau} \ddot{x}^2(0) - 2\lambda \int_0^s dt F_{LD}^2(t). \tag{67}$$

Consequently, we can write for the physical solutions to the LD equation

$$\begin{aligned} -\lambda \frac{d}{ds} (e^{-s/\lambda} \ddot{x}^2) &= \left[\ddot{x}^2(0) - \lambda \frac{d}{d\tau} \ddot{x}^2(0) - 2\lambda \int_0^s dt F_{LD}^2(t) \right] e^{-s/\lambda} \\ \Rightarrow \ddot{x}^2(\tau) &= \ddot{x}^2(0) - \lambda \frac{d}{d\tau} \ddot{x}^2(0) - 2 \int_0^\infty ds e^{-s/\lambda} \int_0^{s+\tau} dt F_{LD}^2(t). \end{aligned} \tag{68}$$

In order to get the latter expression, we have integrated the first equation over s in the limits $[\tau, +\infty)$. The last term on the right-hand side of the second equation in (68) is nonnegative at $\tau \geq 0$ and is zero only for a hyperbolic motion, when $F_{LD}^\mu = 0$. Therefore, the total radiation power of a charged particle moving in a constant electromagnetic field does not grow (this feature was mentioned in [40], but without a rigorous proof) and remains constant only for the hyperbolic motion.

It is instructive to obtain this result in a different way. Introducing for brevity the notation, $f_\mu := F_{\mu\nu} \dot{x}^\nu$, and convolving the LD equation (5) with $F_{\mu\nu} f^\nu$, we find

$$f^\mu \dot{f}_\mu = \lambda (f^\mu \ddot{f}_\mu + \dot{x}^2 f^2) \Rightarrow f^\mu \dot{f}_\mu - \lambda (f^\mu \dot{f}_\mu)^\cdot = \lambda (\dot{x}^2 f^2 - \dot{f}^2). \tag{69}$$

Whence it follows for the physical solutions

$$f^\mu \dot{f}_\mu = \int_\tau^\infty ds e^{(\tau-s)/\lambda} (\dot{x}^2 f^2 - \dot{f}^2) \geq 0, \tag{70}$$

where the functions entering the integrand are taken at the proper-time s . The last inequality is a consequence of the fact that the expression standing in the round brackets is always nonnegative as one can easily check in the momentary comoving frame. Thus the Lorentz force squared is a nondecreasing function of the proper-time. On the other hand, convolving the LD equation (5) with the Lorentz force, we get

$$f^\mu \ddot{x}_\mu - \lambda (f^\mu \ddot{x}_\mu)^\cdot = f^2 \Rightarrow f^\mu \ddot{x}_\mu = \int_0^\infty \frac{dt}{\lambda} e^{-t/\lambda} f^2(\tau + t). \tag{71}$$

The left-hand side of the latter equation can be rewritten with the help of the LD equation as

$$f^\mu \ddot{x}_\mu = (\ddot{x}^\mu - \lambda F_{LD}^\mu) \ddot{x}_\mu = \ddot{x}^2 - \frac{\lambda}{2} (\ddot{x}^2)^\cdot. \tag{72}$$

Then, using the standard trick, we come to

$$\begin{aligned} -\ddot{x}^2 &= -2 \int_0^\infty \frac{ds}{\lambda} \int_0^\infty \frac{dt}{\lambda} e^{-(2s+t)/\lambda} f^2(\tau + s + t) \\ &\leq -2f^2(\tau) \int_0^\infty \frac{ds}{\lambda} \int_0^\infty \frac{dt}{\lambda} e^{-(2s+t)/\lambda} = -f^2(\tau), \end{aligned} \tag{73}$$

for the physical solutions. From the first equality we see again that the total radiation power does not grow with time. The last inequality says that the total radiation power calculated by using the LD equation is less than the same quantity calculated with the help of the Lorentz equation in the same field.

Taking into account the above analysis of the particle motion in a constant electromagnetic field, we deduce that for $\omega_1 = 0$ (see (9)) the total radiation power decreases monotonically to zero, while for $\omega_1 \neq 0$ it declines monotonically to the constant value

$$\mathcal{P} = -\lambda\ddot{x}^2 = \lambda\omega_1^2, \tag{74}$$

which corresponds to a hyperbolic motion. A charged particle tends to move along the trajectory with a minimum radiation. Curiously, this property of the evolution of charged particles complies with a general principle of the least entropy production for non-equilibrium systems [24,25].

As far as the nonuniform electromagnetic fields of the form (28) are concerned, it is hard to prove a monotone decrease of the radiation power with time. Apparently, this property does not even hold in general. Nevertheless, the second property that the radiation reaction force reduces the radiation power can be deduced. Differentiating the expression (35) written in terms of b and ρ with respect to the proper-time and employing the LD equation (37), we have

$$(\ddot{x}^2)' = \frac{2}{\lambda}(\ddot{x}^2 + \rho v_-^2 \omega \cos \psi). \tag{75}$$

This yields for the physical solutions

$$-\ddot{x}^2 = 2 \int_{\tau}^{\infty} \frac{ds}{\lambda} e^{2(s-\tau)/\lambda} \rho v_-^2 \omega \cos \psi, \tag{76}$$

where the subintegral functions depend on the proper-time s . When the amplitude of the electromagnetic wave is the constant ω_0 , we arrive at the estimate

$$-\ddot{x}^2 \leq 2\omega_0 \int_{\tau}^{\infty} \frac{ds}{\lambda} e^{2(s-\tau)/\lambda} \rho v_-^2 \leq \omega_0^2 v_-^2(\tau) \leq \omega_0^2 v_-^2(0), \tag{77}$$

where we have used the inequality (39) and taken into account that v_- decreases monotonically (see Eqs. (34)). The penultimate inequality in formula (77) says that the total radiation power of a charged particle moving in a circularly polarized plane electromagnetic wave of constant amplitude does not exceed the total radiation power calculated by using the Lorentz equations for the same field configuration. In this case, the inequality (77) bounds the total radiation power from above by a monotonically decreasing function proportional to the square of the Lorentz force. In other words, the radiation reaction results in a lessening of the total radiation power as in the case of constant external electromagnetic fields. The same property was pointed out in [8] for a circularly polarized wave propagating along a constant electric field, where it is possible to find the exact stationary solution [41] to the LD equation. It is appealing to give a proof of this property for a general external electromagnetic field or to find any counterexample. Note that for a linearly polarized plane wave of constant amplitude the inequality (77) entails a weaker limitation on the total radiation power. In this case, it is bounded from above by a monotonically decreasing function proportional to the maximum of the square of the Lorentz force over the wave period.

5. Spectral density of radiation

5.1. Constant homogeneous crossed fields

The spectral density of the radiation power of a charged particle moving along the worldline $x^\mu(\tau)$ is given by the standard formula [29]

$$d\mathcal{E}(\mathbf{k}) = -e^2 j_\mu^*(k) j^\mu(k) \frac{d\mathbf{k}}{4\pi^2}, \quad k^2 = k_+ k_- - k_1^2 - k_3^2 = 0, \tag{78}$$

where

$$j_\mu(k) := \int dx_\mu(\tau) e^{-ik_\nu x^\nu(\tau)}, \quad j_\mu^*(k) j^\mu(k) = \text{Re}(j_-^* j_+) - |j_1|^2 - |j_3|^2. \tag{79}$$

Therefore, in order to find the spectral density of radiation, it is sufficient to obtain an expression for the Fourier transform of the current density. In the case at hand, i.e., for a charged particle moving in a crossed electromagnetic field, $j_3 = 0$ in the system of coordinates associated with the tetrad. On integrating the asymptotics (56), one derives

$$k_\mu x^\mu = \sqrt{\frac{2}{\lambda}} \frac{k_-}{10} \left(|\omega| \tau^{5/2} - \frac{10}{3} \zeta' \tau^{3/2} + 5|\zeta|^2 \frac{\tau^{1/2}}{|\omega|} \right), \tag{80}$$

where we have introduced a useful notation

$$\zeta \equiv \zeta' + i\zeta'' \equiv |\zeta| e^{i\varphi} := \frac{\text{sgn}(\omega)k_1 + i|k_3|}{k_-}. \tag{81}$$

The complex number ζ does not depend on the energy k_0 of the radiated photon and defines uniquely the exit angle of this photon.

Let us start with the Fourier transform of the current density component j_1 . Other components are calculated in a similar way. Substituting (56)–(79) and introducing the new integration variable

$$y^2 := |\omega| \tau, \tag{82}$$

we have

$$j_1 = \sqrt{\frac{2}{\lambda}} \frac{\text{sgn}(\omega)}{|\omega|^{3/2}} \int_0^\infty dy y^2 e^{-is}, \quad S := \sqrt{\frac{2}{\lambda}} \frac{k_- y}{10|\omega|^{3/2}} \left(y^4 - \frac{10}{3} \zeta' y^2 + 5|\zeta|^2 \right). \tag{83}$$

The lower integration limit $y = 0$ corresponds to that value of the proper-time when the asymptotic expression (19) becomes valid with a high accuracy. We shall find below the condition that guarantees an independence (with a good accuracy) of the spectral density of radiation from the concrete choice of this instant of time. It is a consequence of the fact that the main contribution to the integral (83) comes from a small neighbourhood of the saddle point of the function $S(y)$.

The behaviour of the function $S(y)$ in the complex y plane is depicted on Fig. 1. Since

$$S'(y) = \sqrt{\frac{2}{\lambda}} \frac{k_-}{2|\omega|^{3/2}} (y^2 - \zeta)(y^2 - \zeta^*), \tag{84}$$

the four saddle points arranged symmetrically with respect to the origin of the y plane merge into two points when ζ is real (i.e., at $k_3 = 0$). The main contribution to the integral comes from the saddle point $y = \sqrt{\zeta^*}$, this saddle point becoming degenerate when ζ is real. The optimum integration contour is shown in Fig. 1, where we have also introduced the upper integration limit y_r in order to estimate the radiation formation time (see, e.g., [42,43]). A concrete value of the upper integration limit is found from the requirement that, in increasing y_r , the contribution to the spectral density of radiation from the contour $[y_r, +\infty)$ becomes negligibly small as compared with the contribution from the contour $[0, y_r]$.

It is useful for a further analysis to distinguish three cases

$$(i) (20\alpha)^{2/5} |\zeta'| \gg 1, \quad (ii) (20\alpha)^{2/5} |\zeta''| \gg 1, \quad (iii) (20\alpha)^{2/5} |\zeta| \ll 1, \tag{85}$$

where

$$\alpha := \sqrt{\frac{2}{\lambda}} \frac{k_-}{10|\omega|^{3/2}}. \tag{86}$$

These three conditions ought to be considered as the restrictions on the possible observation angles and the energies of photons. In the last case (iii), as we shall show, the integral is saturated in the vicinity of the point $y = 0$ and, consequently, it depends severely on the evolution of a charged particle before the moment when this particle passes to the asymptotic regime (56).

We proceed with the first case. Introduce the new integration variable

$$t := (20\alpha\zeta')^{1/3} (y - \sqrt{\zeta'}). \tag{87}$$

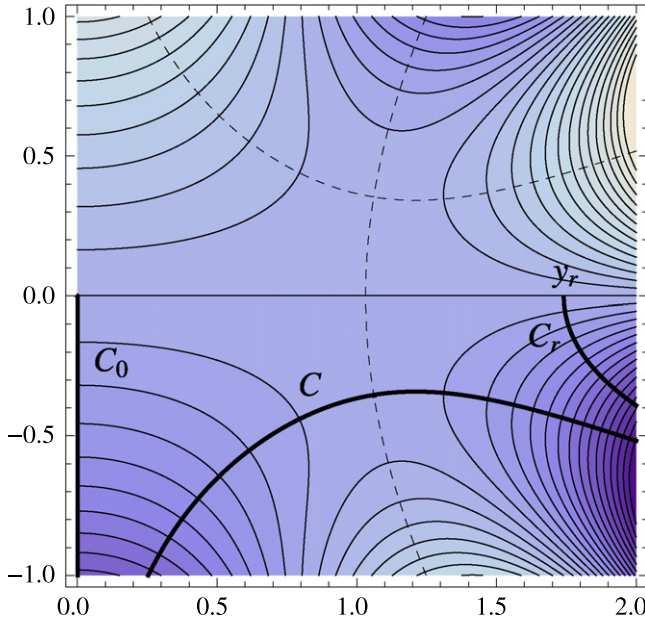


Fig. 1. The level lines of the function $\text{Im } S$ and the integration contour at $\zeta = 1 + 3i/4$. The initial integration contour $[0, y_r]$ is deformed to the contour $C_0 \cup C \cup C_r$. The lines of the steepest descent are depicted as the dashed lines on the figure. One of these lines coincides with the contour C . The steepest descent lines intersect at the saddle points.

Then

$$S = \alpha \left[\frac{8}{9} (\zeta')^{5/2} + 5(\zeta'')^2 \zeta'^{1/2} \right] + Bt + \frac{t^3}{3} + \frac{ht^4}{4} + \frac{h^2 t^5}{20}, \tag{88}$$

where

$$B \equiv bk_3^2 := 5 \left(\frac{\alpha^2}{20\zeta'} \right)^{1/3} (\zeta'')^2 = \frac{2k_3^2}{\lambda|\omega|^3 (20\alpha)^{4/3} \zeta'^{1/3}} = \frac{k_3^2}{2|\omega|k_- (\lambda \text{sgn}(\omega)k_1)^{1/3}}, \tag{89}$$

$$h := (20\alpha)^{-1/3} (\zeta')^{-5/6}.$$

Also we need

$$y^2 = \zeta'(1 + 2ht + h^2 t^2), \quad y^4 = \zeta'^2(1 + 4ht + 6h^2 t^2 + 4h^3 t^3 + h^4 t^4),$$

$$\int \frac{dt}{2\pi} e^{-i(Bt+t^3/3)} =: \text{Ai}(B), \quad \int \frac{dt}{2\pi} t e^{-i(Bt+t^3/3)} =: i \text{Ai}'(B). \tag{90}$$

Notice that $h \ll 1$ in the case (i). We shall carry out all the calculations taking into account the terms up to the order of h^2 . It will turn out that only at the order h^2 does the first non-vanishing contribution to the spectral density of radiation arise. Substituting (88) and (90) to the integral (83) taken on the contour C (see Fig. 1) and developing the expression as a series in h , we obtain

$$j_1 \approx \text{sgn}(\omega)c \left[I_0 + h \left(2I_1 - \frac{iI_4}{4} \right) + h^2 \left(I_2 - \frac{I_8}{32} - \frac{11i}{20} I_5 \right) \right], \tag{91}$$

where

$$c := \frac{(20\alpha\zeta')^{2/3}}{2k_-} e^{-i\alpha \left(\frac{8}{9} (\zeta')^{5/2} + 5(\zeta'')^2 \zeta'^{1/2} \right)}, \quad I_n := \int_C dt t^n e^{-i(Bt+t^3/3)}. \tag{92}$$

Of course, the contours C_0 and C_r also contribute to j_1 . The magnitude of these contributions will be estimated below, but now we assume that such contributions are small.

Similar calculations for the other current density components

$$j_- = \sqrt{\frac{2}{\lambda}} |\omega|^{-3/2} \int_0^\infty dy e^{-is}, \quad j_+ = \sqrt{\frac{2}{\lambda}} |\omega|^{-3/2} \int_0^\infty dy y^4 e^{-is}, \tag{93}$$

yield

$$j_- \approx \frac{c}{\zeta'} \left[I_0 - ih \frac{I_4}{4} - h^2 \left(\frac{I_8}{32} + i \frac{I_5}{20} \right) \right], \tag{94}$$

$$j_+ \approx c \zeta' \left[I_0 + h \left(4I_1 - i \frac{I_4}{4} \right) + h^2 \left(6I_2 - \frac{I_8}{32} - \frac{21i}{20} I_5 \right) \right].$$

One may verify that the current density is still conserving up to the order h^2 . Indeed, the equality,

$$k_- j_+ + k_+ j_- - 2k_1 j_1 = 0, \tag{95}$$

is equivalent to

$$\left(\frac{\zeta''}{\zeta'} \right)^2 \left[I_0 - ih \frac{I_4}{4} - h^2 \left(\frac{I_8}{32} + i \frac{I_5}{20} \right) \right] + 4h^2 I_2 = 0. \tag{96}$$

Taking into account that

$$\left(\frac{\zeta''}{\zeta'} \right)^2 = 4Bh^2, \quad I_2(B) = -BI_0(B), \tag{97}$$

we ascertain that the charge conservation law is fulfilled. This provides an indirect test of the correctness of the expressions for the Fourier transformed current density.

Substituting the expressions obtained to (79) and keeping the terms up to the order of h^2 , we have

$$j_\mu^*(k) j^\mu(k) \approx 4|c|^2 |h|^2 (\text{Re}(I_0^* I_2) - |I_1|^2) = -4\pi hc^2 (|B| |Ai(B)|^2 + |Ai'(B)|^2), \tag{98}$$

$$|hc|^2 = |b| e^{2\omega \text{Im} \left[\frac{8}{3} (\zeta')^{3/2} + 5\zeta'^{2} \zeta'^{1/2} \right]} = |b| \exp \left[-\sqrt{\frac{2}{\lambda}} \frac{|k_1|^{1/2}}{|k_- \omega|^{3/2}} \left(\frac{8}{15} k_1^2 + k_3^2 \right) \theta(-\omega k_1) \right].$$

Observe that the expression above is symmetric with respect to a sign change of k_3 , as it should be. Also we see that, at $\text{sgn}(\omega k_1) < 0$, this contribution to the spectral density of radiation is exponentially suppressed. This is a consequence of the fact that the most part of the energy of the electromagnetic waves is radiated along the 3-velocity of a charged particle (the spotlight effect), while the exit direction of the photon and the direction of the 3-velocity of the particle are always in different hemispheres for $\text{sgn}(\omega k_1) < 0$.

At $\text{sgn}(\omega k_1) \geq 0$ (cf. [43,44]),

$$j_\mu^*(k) j^\mu(k) \approx -(4\pi)^2 b \{ bk_3^2 [Ai(bk_3^2)]^2 + [Ai'(bk_3^2)]^2 \}, \tag{99}$$

where we have shown the dependence on k_3 explicitly. The function,

$$B[Ai(B)]^2 + [Ai'(B)]^2 = \frac{3^{-2/3}}{\Gamma^2(1/3)} + \frac{3^{2/3} B}{\Gamma^2(-1/3)} - \frac{\sqrt{3}}{2\pi} B^2 + O(B^3)$$

$$= e^{-\frac{4}{3} B^{3/2}} \left(\frac{B^{1/2}}{2\pi} + O(1/B) \right), \tag{100}$$

possesses the extremum at the point $B_{\text{ext}} \approx 0.286$ and

$$\frac{\{B[Ai(B)]^2 + [Ai'(B)]^2\}_{B=B_{\text{ext}}}}{\{B[Ai(B)]^2 + [Ai'(B)]^2\}_{B=0}} \approx 1.25. \tag{101}$$

To put in another way, a charged particle emits a maximum number of photons with a given energy not in the orbit plane (as one may expect for an ultrarelativistic particle), but at the angle specified by the relation

$$\frac{k_3^2}{k_0^2} = B_{\text{ext}} \frac{2|\omega|k_-(\lambda|k_1|)^{1/3}}{k_0^2}. \tag{102}$$

The expression (99) as a function of the photon energy k_0 is monotonically decreasing and singular at $k_0 = 0$ since $B \sim k_0^{2/3}$ and $b \sim k_0^{-4/3}$. At the small photon energies, the condition (i) in (85) is violated and the expression (99) becomes invalid. Transforming the integration measure in the momentum space, the spectral density of radiation can be cast into the form

$$d\mathcal{E} \approx 4e^2 B \{ B[\text{Ai}(B)]^2 + [\text{Ai}'(B)]^2 \} \frac{k_0^2}{k_3^2} dk_0 d\Omega, \tag{103}$$

where $d\Omega$ is the element of a solid angle. The function

$$B \{ B[\text{Ai}(B)]^2 + [\text{Ai}'(B)]^2 \} \tag{104}$$

possesses the extremum at the point $B'_{\text{ext}} \approx 0.8$, which determines the energy of photons,

$$k_{\text{ext}}^0 = (\lambda|k_1|/k_0)^{1/2} \frac{(2B'_{\text{ext}}|\omega|k_-/k_0)^{3/2}}{(k_3/k_0)^3}, \tag{105}$$

where the maximum of radiation occurs at a given exit angle.

Now, let the condition (i) in (85) be violated, but the inequality (ii) be satisfied. Then the saddle points $y = \sqrt{\zeta}$ and $y = \sqrt{\zeta^*}$ are well separated and the standard WKB method is applicable. In the extremum point $y = \sqrt{\zeta^*}$, we have

$$S'' = -20\alpha y(y^2 - \zeta') = 20i\alpha \zeta'' \sqrt{\zeta^*}. \tag{106}$$

Using the standard WKB formulas, we derive in the leading order

$$j_1 \approx \left(\sqrt{\frac{2}{\lambda}} \frac{\pi \zeta^*}{|k_3| |\omega|^{3/2}} \right)^{1/2} \exp \left[-i \frac{10}{3} \alpha \sqrt{\zeta^*} \left(|\zeta|^2 - \frac{1}{5} \zeta^{*2} \right) \right]. \tag{107}$$

It is easy to verify that the real part of the expression standing in the exponent is negative and so this contribution is exponentially suppressed. The analogous calculations for the rest current density components result in

$$j_- \approx \frac{j_1}{\zeta^*}, \quad j_+ \approx \zeta^* j_1, \tag{108}$$

in the leading order. Hence, we obtain

$$\begin{aligned} j_\mu^*(k)j^\mu(k) &\approx \left(\text{Re} \frac{\zeta^*}{\zeta} - 1 \right) |j_1|^2 = -2 \frac{\zeta''^2}{|\zeta|^2} |j_1|^2 \\ &= -\sqrt{\frac{2}{\lambda}} \frac{2\pi \sin \varphi}{k_- |\omega|^{3/2}} \exp \left[-\frac{20}{3} \alpha |\zeta|^{5/2} \left(\sin \frac{\varphi}{2} - \frac{1}{5} \sin \frac{5\varphi}{2} \right) \right] \\ &= -\frac{2\pi \sin \varphi}{5\alpha \lambda |\omega|^3} \exp \left[-\frac{20}{3} \alpha |\zeta|^{5/2} \left(\sin \frac{\varphi}{2} - \frac{1}{5} \sin \frac{5\varphi}{2} \right) \right], \end{aligned} \tag{109}$$

where, recall, $\varphi \in [0, \pi]$ is the phase of ζ .

Consider the case (iii) in (85). If this condition is fulfilled then, redefining the integration variable $y \rightarrow \alpha^{-1/5}y$, it is not difficult to see that the last two terms in the expression standing in the exponent

(83) are small. These terms can be taken into account by developing the exponent as a Taylor series in the small parameter and, subsequently, by a termwise integration of the series obtained. The integrals arising in the course of this procedure are reduced to the gamma functions and depend appreciably on the lower integration limit $y = 0$. The leading contribution appears to be

$$j_{\mu}^*(k)j^{\mu}(k) \approx \frac{2}{25\lambda|\omega|^3\alpha^{6/5}} \left[\frac{\sqrt{5}-1}{4} \Gamma(1/5) - \Gamma^2(3/5) \right]. \tag{110}$$

However, this expression may change substantially when the contribution of particle's trajectory before the instant $\tau = 0$ is taken into account.

Let us estimate the contributions to the spectral density of radiation from the contours C_0 and C_r . For the contour C_0 , at

$$\alpha^{1/2}|\zeta| \gg 1, \tag{111}$$

the integral is saturated near the boundary point $y = 0$ of the contour. The condition (111) is rather close to the first two conditions in (85). The contribution from the boundary point is given by the standard expression and in the leading order takes the form

$$j_1^{C_0} = -\sqrt{\frac{2}{\lambda}} \frac{\text{sgn}(\omega)}{|\omega|^{3/2}} \int_{-\infty}^0 dy y^2 e^{-iS} \approx i \text{sgn}(\omega) \sqrt{\frac{2}{\lambda}} \frac{2(5\alpha|\zeta|^2)^{-3}}{|\omega|^{3/2}} = \text{sgn}(\omega) \frac{8i\lambda|\omega|^3}{k_-^3|\zeta|^6}. \tag{112}$$

In a similar way,

$$\begin{aligned} j_+^{C_0} &= -\sqrt{\frac{2}{\lambda}} |\omega|^{-3/2} \int_{-\infty}^0 dy y^4 e^{-iS} \approx -i\sqrt{\frac{2}{\lambda}} \frac{24(5\alpha|\zeta|^2)^{-5}}{|\omega|^{3/2}} = -\frac{192i\lambda^2\omega^6}{k_-^5|\zeta|^{10}}, \\ j_-^{C_0} &= -\sqrt{\frac{2}{\lambda}} |\omega|^{-3/2} \int_{-\infty}^0 dy e^{-iS} \approx -i\sqrt{\frac{2}{\lambda}} \frac{(5\alpha|\zeta|^2)^{-1}}{|\omega|^{3/2}} = -\frac{2i}{k_-|\zeta|^2}. \end{aligned} \tag{113}$$

Therefore, the contour C_0 makes the contribution to the spectral density of radiation proportional to

$$[j_{\mu}^*(k)j^{\mu}(k)]_{C_0} = [\text{Re}(j_{-+}^*) - |j_1|^2]_{C_0} \approx \frac{40}{\lambda|\omega|^3} (5\alpha|\zeta|^2)^{-6}. \tag{114}$$

In the case when this contribution is comparable with or larger than the contributions calculated above for the cases (i) and (ii), the spectral density of radiation depends severely on the evolution of a charged particle before the instant when this particle reaches the asymptotic regime.

For the contour C_r we assume that

$$y_r^2 = |\omega|\tau_r \gg |\zeta|. \tag{115}$$

This means that the point y_r is much more right than the saddle point $y = \sqrt{\zeta^*}$ (see Fig. 1) and so the integral is saturated. Then

$$\begin{aligned} j_1^{C_r} &\approx \sqrt{\frac{2}{\lambda}} \frac{\text{sgn}(\omega)}{|\omega|^{3/2}} \int_{-\infty}^{y_r} dy y^2 e^{-i\alpha y^5} = -\sqrt{\frac{2}{\lambda}} \frac{\text{sgn}(\omega)}{|\omega|^{3/2}} \frac{\Gamma(3/5, i\alpha y_r^5)}{5(i\alpha)^{3/5}}, \\ j_-^{C_r} &\approx \sqrt{\frac{2}{\lambda}} |\omega|^{-3/2} \int_{-\infty}^{y_r} dy e^{-i\alpha y^5} = -\sqrt{\frac{2}{\lambda}} \frac{\Gamma(1/5, i\alpha y_r^5)}{5(i\alpha)^{1/5}|\omega|^{3/2}}, \\ j_+^{C_r} &\approx \sqrt{\frac{2}{\lambda}} |\omega|^{-3/2} \int_{-\infty}^{y_r} dy y^4 e^{-i\alpha y^5} = -\sqrt{\frac{2}{\lambda}} \frac{\Gamma(1, i\alpha y_r^5)}{5i\alpha|\omega|^{3/2}}. \end{aligned} \tag{116}$$

In the cases (i) and (ii) (see (85)) and when (115) is satisfied, the arguments of the incomplete gamma function are large and so we can employ its asymptotic expansion at large arguments [45] with the

result

$$\begin{aligned}
 [j_{\mu}^{*}(k)j^{\mu}(k)]_{C_r} &= [\text{Re}(j_{-}^{*}j_{+}) - |j_1|^2]_{C_r} \\
 &\approx \frac{2}{25\lambda|\omega|^3\alpha^{6/5}} \left[-\frac{12}{25}(\alpha y_r^5)^{-14/5} + O((\alpha y_r^5)^{-24/5}) \right] \\
 &\approx -\frac{24}{625\lambda|\omega|^3\alpha^4 y_r^{14}}.
 \end{aligned}
 \tag{117}$$

This contribution is to be compared with the main contribution for the cases (i) and (ii).

In the case (i) at $\text{sgn}(\omega k_1) \geq 0$, the main contribution (99) dominates when (we use for the estimate the value of the spectral density of radiation at $B = 0$)

$$\begin{aligned}
 (20\alpha)^{2/5}|\zeta| &\gg \left(\frac{\pi^2}{32^{2/3}5120\Gamma^2(1/3)} \right)^{-3/35} \approx 2.15, \\
 (20\alpha)^{2/5}|\zeta'| &\gg 3^{1/4} \left(\frac{\pi}{8\Gamma(1/3)} \right)^{-3/10} \approx 2.34.
 \end{aligned}
 \tag{118}$$

The first inequality is obtained from comparison of (99) with (114), while the second inequality follows from comparison of (99) with (117) and taking into account the inequality (115). As we see, in the case (i) at $\text{sgn}(\omega k_1) \geq 0$, the main contribution to the spectral density of radiation is given by (99) provided that the radiation has a time to be formed, viz., the inequality (115) holds.

Consider the case (i) at $\text{sgn}(\omega k_1) < 0$. To make the estimates we assume that $\zeta'' = 0$, otherwise at $\zeta'' \gg |\zeta'|$ we fall into the case (ii). Keeping this in mind, we deduce that (98) makes the main contribution to the spectral density of radiation when

$$\begin{aligned}
 \frac{5^{11/14}}{6^{1/7}} \left(\frac{\pi}{\Gamma(1/3)} \right)^{3/7} \alpha |\zeta'|^{5/2} e^{-\frac{8}{7}\alpha|\zeta'|^{5/2}} &\approx 2.94\alpha |\zeta'|^{5/2} e^{-\frac{8}{7}\alpha|\zeta'|^{5/2}} \gg 1, \\
 \frac{5}{2^{1/4}3^{5/8}} \left(\frac{\pi}{\Gamma(1/3)} \right)^{3/4} \alpha |\zeta'|^{5/2} e^{-2\alpha|\zeta'|^{5/2}} &\approx 2.38\alpha |\zeta'|^{5/2} e^{-2\alpha|\zeta'|^{5/2}} \gg 1,
 \end{aligned}
 \tag{119}$$

where the first condition follows from comparison of (98) with (114) and the second inequality is obtained from comparison of (98) with (117) taking into account the inequality (115). Both the inequalities cannot be satisfied as long as the expressions entering their left-hand sides are less than unity. Consequently, the expression (98) should not be used for the evaluation of the spectral density of radiation when $\text{sgn}(\omega k_1) < 0$.

In the case (ii), comparing (109) with (114) and (117), we see that the WKB answer (109) gives the major contribution when

$$\begin{aligned}
 \frac{625}{16\,384} \frac{\pi}{\sqrt{2|\zeta|}} (\alpha(2|\zeta|)^{5/2})^5 e^{-\alpha(2|\zeta|)^{5/2}} &\gg 1, \\
 \frac{125}{1536} \frac{\pi}{\sqrt{2|\zeta|}} (\alpha(2|\zeta|)^{5/2})^3 e^{-\alpha(2|\zeta|)^{5/2}} &\gg 1,
 \end{aligned}
 \tag{120}$$

respectively. To obtain these estimates, we have assumed that $\zeta'' \gg |\zeta'|$ and the condition (115) has been used in the second inequality. Notice that both of these inequalities can be satisfied only if $|\zeta|$ is rather small.

5.1.1. Experimental verification

The contemporary experimental facilities allow one to measure, in principle, the radiation formed on the asymptotics of the physical solutions to the LD equation in the crossed fields. To this aim the intensity of a laser radiation has to be increased by the three orders of magnitude as against one given in (7). Such intensities will be accessible in the near future [1]. To observe the radiation, which we

have described above, one should launch the bunch of electrons to the plane electromagnetic wave in such a way that all the time, when the electrons are in the electromagnetic wave, the strength of the electromagnetic field in the vicinity of the electron bunch must change negligibly. This is possible for the ultrarelativistic electrons scattering on the laser photons at an obtuse angle, when the diameter of the laser beam is of the order of a few wavelengths and the characteristic size of the electron bunch is much lesser than the wavelength of the incident photons. Moreover, to conduct a series of experiments with the same strength of the electromagnetic field in the crossing region of the electron and laser beams, these beams have to be synchronized.

Let N be the number of waves that are traversed by the electron moving in the electromagnetic wave, $N \ll 1$. Then the proper-time τ_{esc} , which the ultrarelativistic electron spends in the electromagnetic wave, is obtained from the simple system of equations

$$\gamma u \tau_{esc} \sin \alpha = d, \quad \gamma(1 + u \cos \alpha) \tau_{esc} = \lambda_\gamma N, \quad \lambda_\gamma := 2\pi \Omega^{-1}, \tag{121}$$

where u is a module of the electron velocity, α is the entrance angle of the electron to the laser beam, which is counted from the electromagnetic wave propagation axis, and d is the diameter of the laser beam. All the quantities are taken in the laboratory frame. If

$$\gamma^{-2} \ll \left(N \frac{\lambda_\gamma}{d} \right)^2 \ll u^2, \tag{122}$$

then one can put $u = 1$ in formulas (121) and, in what follows, we assume this is so. The time needed for a charged particle to go to the asymptotic regime is determined by the two characteristic times [5]

$$\tau_{1c} := (2\lambda\omega^2 v_-(0))^{-1}, \quad \tau_{2c} := \frac{|v_1(0)|}{v_-(0)|\omega|}. \tag{123}$$

The requirement that these characteristic times are much lesser than τ_{esc} leads to the inequalities

$$\left(\frac{d}{\lambda_\gamma} \right)^2 \ll 8\pi N^3 \frac{\lambda\omega^2 \gamma}{\Omega}, \quad \frac{d}{\lambda_\gamma} \gg \frac{\gamma \Omega}{\pi |\omega|}, \tag{124}$$

respectively. Taking $\omega = 4.65 \cdot 10^{-3}$, that is greater by one and a half order than in (7), $\gamma = 10^3$, and $N = 1/5$, we have

$$\left(\frac{d}{\lambda_\gamma} \right)^2 \ll 11, \quad \frac{d}{\lambda_\gamma} \gg 0.13, \tag{125}$$

and the entrance angle $\alpha \approx 168.5^\circ$. These inequalities can be satisfied when the diameter of the laser beam is of the order of two wavelengths.

The requirement that the radiation has a time to be formed on the asymptotics, inequality (115) at $\tau_r = \tau_{esc}$, transforms into

$$|\zeta| \ll \frac{4\pi |\omega|}{\gamma N \Omega} \approx 149. \tag{126}$$

This results in (reasonable) constraints on the observation angles. The condition (i) in (85) is convenient to rewrite as

$$k_0 \gg \left[\frac{\lambda\omega^3 (k_-/k_0)^3}{8(k_1/k_0)^5} \right]^{1/2} \approx \frac{(k_-/k_0)^{3/2}}{(k_1/k_0)^{5/2}} 4 \text{ eV}. \tag{127}$$

As we saw, the expression (103) representing the main contribution to the spectral density of radiation formed on the asymptotics possesses the maximum at $B = B_{ext}$ as a function of the photon energy k_0 . The maximum spectral density of the radiation formed can be observed when the extremum point lies

in the photon energy domain satisfying the inequality (127). The condition (127) restricts the possible values of the parameter B as

$$\frac{k_3^2}{4k_1^2} \ll B. \tag{128}$$

Hence, the maximum of the radiation formed on the asymptotics can be observed if

$$\frac{k_3^2}{4k_1^2} \ll B'_{\text{ext}}. \tag{129}$$

This maximum lies in the region of a few dozens of eVs,

$$k_{\text{ext}}^0 \approx (|k_1|/k_0)^{1/2} \frac{(k_-/k_0)^{3/2}}{(k_3/k_0)^3} 22.9 \text{ eV}, \tag{130}$$

and corresponds to soft photons. In this energy region, the radiation is well described by classical theory [9–11] and, therefore, we may expect that the expressions for the spectral density of radiation obtained above agree with the one observable in experiments with a high degree of accuracy.

5.2. Plane electromagnetic wave

Let us derive the expressions for the spectral density of radiation formed on the asymptotics of the physical solutions to the LD equation in a plane electromagnetic wave of constant amplitude. We shall not elaborate here a thorough analysis as in the case of the crossed fields (it will be given elsewhere) and shall obtain only the main contribution to the spectral density of radiation formed.

5.2.1. Linearly polarized wave

Consider, at first, the radiation of a charged particle in a linearly polarized plane electromagnetic wave. The asymptotics of the physical solutions to the LD equation was found by us in (49). The corrections to this asymptotics are of the order of $(\Omega x_-)^{-1}$ as compared with the main contribution and negligible provided the electron traverses at least two “humps” of the electromagnetic wave. Henceforward, we change the notation $\omega_0 \rightarrow \omega$ for brevity. Then, integrating the asymptotics (49) and casting out the terms of the order of $(\Omega x_-)^{-1}$, we arrive at

$$x_- = 2 \left(\frac{\tau}{\lambda \omega^2} \right)^{1/2}, \quad x_+ = \frac{\lambda^2 \omega^4 x_-^3}{12} \left(1 + \frac{\omega^2}{2\Omega^2} \right), \quad x_1 = -\frac{\lambda \omega^3}{2\Omega^2} x_- \cos \psi. \tag{131}$$

In general, the analysis of the spectral density of radiation is rather similar to the procedure used in deriving the spectral density of radiation on the solutions to the Lorentz equation in the electromagnetic wave [43,46]. It is useful to change the integration variable in the integrals (79) and introduce the variable y such that

$$x_- = 2y \left[\lambda^2 \omega^4 \left(1 + \frac{\omega^2}{2\Omega^2} \right) k_- \right]^{-1/3}. \tag{132}$$

As a result, we have

$$\begin{aligned} k_\mu x^\mu &= \frac{y^3}{3} + y \left(k_+ + \frac{\lambda \omega^3 k_1}{\Omega^2} \cos \psi \right) \left[\lambda^2 \omega^4 \left(1 + \frac{\omega^2}{2\Omega^2} \right) k_- \right]^{-1/3} \\ &=: \frac{y^3}{3} + By + ay \cos \psi \end{aligned} \tag{133}$$

with

$$\psi = \tilde{\Omega} y + \psi_0, \quad \tilde{\Omega} := 2\Omega \left[\lambda^2 \omega^4 \left(1 + \frac{\omega^2}{2\Omega^2} \right) k_- \right]^{-1/3}, \tag{134}$$

and

$$\begin{aligned}
 j_- &= 2 \left[\lambda^2 \omega^4 \left(1 + \frac{\omega^2}{2\Omega^2} \right) k_- \right]^{-1/3} \int_0^\infty dy e^{-ik_\mu x^\mu}, \\
 j_+ &= \frac{2}{k_- (1 + \omega^2/2\Omega^2)} \int_0^\infty dy y^2 \left(1 + \frac{\omega^2}{\Omega^2} \sin^2 \psi \right) e^{-ik_\mu x^\mu}, \\
 j_1 &= \frac{2\lambda\omega^3}{\Omega} \left[\lambda^2 \omega^4 \left(1 + \frac{\omega^2}{2\Omega^2} \right) k_- \right]^{-2/3} \int_0^\infty dy y \sin \psi e^{-ik_\mu x^\mu}.
 \end{aligned}
 \tag{135}$$

In the formulas above we have used the approximate expressions (49) for the 4-velocities. These expressions differ from what one would obtain differentiating the asymptotics (131). The reason is that the terms of the order of $(\Omega x_-)^{-1}$ in comparison with the leading contribution were neglected in (131). This will result in an apparent violation of the charge conservation law (95), which is only approximately satisfied now (see below).

We start with the integral defining the Fourier transform of the current density component j_- . Using the representation in the form of the Fourier series

$$e^{-iay \cos \psi} = \sum_{n=-\infty}^\infty \int_{-\pi}^\pi \frac{d\varphi}{2\pi} e^{-i(ay \cos \varphi - n\psi)} \cos n\varphi,
 \tag{136}$$

we can write

$$\int_0^\infty dy e^{-ik_\mu x^\mu} = \sum_{n=-\infty}^\infty e^{in\psi_0} \int_{-\pi}^\pi d\varphi \tilde{\text{Ai}}(B + a \cos \varphi - n\tilde{\Omega}) \cos n\varphi,
 \tag{137}$$

where

$$\tilde{\text{Ai}}(B) := \int_0^\infty \frac{dt}{2\pi} e^{-i(Bt+t^3/3)}.
 \tag{138}$$

If the argument of this function is positive, the main contribution to the integral defining this function comes from the neighbourhood of the point $t = 0$,

$$\tilde{\text{Ai}}(B) \xrightarrow{B \rightarrow +\infty} \frac{i}{2\pi B},
 \tag{139}$$

and, consequently, it substantially depends on the form of the evolution before the moment when the charged particle passes to the asymptotic regime. At the large negative argument, the integral is saturated near the saddle point $t = (-B)^{1/2}$,

$$\tilde{\text{Ai}}(B) \xrightarrow{B \rightarrow -\infty} \text{Ai}(B) = (\pi^2 |B|)^{-1/4} \sin(2|B|^{3/2}/3 + \pi/4) + O(|B|^{-7/4}),
 \tag{140}$$

and weakly depends on the integration limits. We are interested in this last case. From (133) and (7) we see that usually

$$a \ll B.
 \tag{141}$$

Therefore, in order to observe the radiation formed on the asymptotics, it is necessary to demand

$$1 \ll \tilde{\Omega} - B \ll 16\pi^2 B^4,
 \tag{142}$$

i.e., the contributions making by the terms with $n \leq 0$ in the series (137) are negligibly small in comparison with the contributions from $n \geq 1$. Furthermore, already the term with $n = 1$ must depend weakly on the integration limits. The last inequality in (142) follows from comparison of the term at $n = 0$ (see (139)) with the term at $n = 1$. Also we have to impose the constraint that in the

saddle point

$$\Omega x_-^{\text{ext}} = \tilde{\Omega} y^{\text{ext}} = \tilde{\Omega}(\tilde{\Omega} - B)^{1/2} \gg 1, \tag{143}$$

but this condition holds already by virtue of the first inequality in (142).

In the case when the condition (142) is fulfilled, the Fourier transform of the current density component j_- reduces to

$$j_- \approx 2 \left[\lambda^2 \omega^4 \left(1 + \frac{\omega^2}{2\Omega^2} \right) k_- \right]^{-1/3} \sum_{n=1}^{\infty} e^{in\psi_0} \int_{-\pi}^{\pi} d\varphi \text{Ai}(B + a \cos \varphi - n\tilde{\Omega}) \cos n\varphi. \tag{144}$$

Other components of the current density are evaluated in a similar way

$$\begin{aligned} j_+ &\approx \sum_{n=1}^{\infty} \frac{-2e^{in\psi_0}}{k_- (1 + \omega^2/2\Omega^2)} \int_{-\pi}^{\pi} d\varphi \text{Ai}''(B + a \cos \varphi - n\tilde{\Omega}) \\ &\times \left[\left(1 + \frac{\omega^2}{2\Omega^2} \right) \cos n\varphi - \frac{\omega^2}{4\Omega^2} \cos(n+2)\varphi - \frac{\omega^2}{4\Omega^2} \cos(n-2)\varphi \right], \\ j_1 &\approx \frac{\lambda\omega^3}{\Omega} \left[\lambda^2 \omega^4 \left(1 + \frac{\omega^2}{2\Omega^2} \right) k_- \right]^{-2/3} \sum_{n=1}^{\infty} e^{in\psi_0} \int_{-\pi}^{\pi} d\varphi \text{Ai}'(B + a \cos \varphi - n\tilde{\Omega}) \\ &\times \left[\cos(n-1)\varphi - \cos(n+1)\varphi \right]. \end{aligned} \tag{145}$$

Assuming that the different phases ψ_0 are equiprobable in the bunch of electrons, we can average over them with a unit weight. Then we have

$$\begin{aligned} \langle \text{Re}(j_-^* j_+) \rangle &\approx -4\lambda^2 \omega^4 \left[\lambda^2 \omega^4 \left(1 + \frac{\omega^2}{2\Omega^2} \right) k_- \right]^{-4/3} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} d\varphi' \text{Ai}(B + a \cos \varphi' - n\tilde{\Omega}) \\ &\times \cos n\varphi' \int_{-\pi}^{\pi} d\varphi \text{Ai}''(B + a \cos \varphi - n\tilde{\Omega}) \\ &\times \left[\left(1 + \frac{\omega^2}{2\Omega^2} \right) \cos n\varphi - \frac{\omega^2}{4\Omega^2} \cos(n+2)\varphi - \frac{\omega^2}{4\Omega^2} \cos(n-2)\varphi \right], \\ \langle |j_1|^2 \rangle &\approx \frac{\lambda^2 \omega^6}{\Omega^2} \left[\lambda^2 \omega^4 \left(1 + \frac{\omega^2}{2\Omega^2} \right) k_- \right]^{-4/3} \\ &\times \sum_{n=1}^{\infty} \left\{ \int_{-\pi}^{\pi} d\varphi \text{Ai}'(B + a \cos \varphi - n\tilde{\Omega}) \left[\cos(n-1)\varphi - \cos(n+1)\varphi \right] \right\}^2. \end{aligned} \tag{146}$$

This, with the account of (78), gives the spectral density of radiation formed on the asymptotics of the physical solutions to the LD equation for the field of a linearly polarized plane electromagnetic wave.

The expression obtained can be simplified under the assumption that a is small. Expanding the Airy functions in a Taylor series in a , we get the integrals of the form

$$\int_{-\pi}^{\pi} d\varphi \cos^k \varphi \cos n\varphi = 2^{1-k} \pi C_k^{(k-n)/2}, \tag{147}$$

at $k \geq n \geq 0$ and $n+k$ is an even number, otherwise these integrals are equal to zero. Then, retaining only the first correction in a in the expression (146), we come to

$$\begin{aligned} \langle \text{Re}(j_-^* j_+) \rangle &\approx -(2\pi)^2 \lambda^2 \omega^4 \left[\lambda^2 \omega^4 \left(1 + \frac{\omega^2}{2\Omega^2} \right) k_- \right]^{-4/3} \\ &\times a^2 \left\{ \left(1 + \frac{\omega^2}{4\Omega^2} \right) \text{Ai}'(B - \tilde{\Omega}) \text{Ai}'''(B - \tilde{\Omega}) - \frac{\omega^2}{8\Omega^2} [\text{Ai}''(B - 2\tilde{\Omega})]^2 \right\}, \end{aligned}$$

$$\begin{aligned} \langle |j_1|^2 \rangle &\approx (2\pi)^2 \frac{\lambda^2 \omega^6}{\Omega^2} \left[\lambda^2 \omega^4 \left(1 + \frac{\omega^2}{2\Omega^2} \right) k_- \right]^{-4/3} \\ &\times \left\{ [\text{Ai}'(B - \tilde{\Omega})]^2 + \left(\frac{a}{2} \right)^2 [\text{Ai}'(B - \tilde{\Omega})\text{Ai}'''(B - \tilde{\Omega}) + [\text{Ai}''(B - 2\tilde{\Omega})]^2] \right\}. \end{aligned} \tag{148}$$

Taking into account that $\omega/\Omega \gg 1$, we eventually arrive at

$$\begin{aligned} \langle \text{Re}(j_-^* j_+) \rangle - |j_1|^2 &\approx - \left(\frac{2\pi}{\omega} \right)^2 \left(\frac{4\Omega}{\lambda k_-^2} \right)^{2/3} \left\{ [\text{Ai}'(B - \tilde{\Omega})]^2 \right. \\ &\left. + \frac{a^2}{8} [4\text{Ai}'(B - \tilde{\Omega})\text{Ai}'''(B - \tilde{\Omega}) + [\text{Ai}''(B - 2\tilde{\Omega})]^2] \right\}. \end{aligned} \tag{149}$$

The development of the Airy function as a Taylor series in a is rapidly converging provided that

$$a^2 \tilde{\Omega} \ll 1. \tag{150}$$

This inequality leads to the additional constraint to (142) on the domain of applicability of the formula (149). The inequality (150) readily follows from the form of the asymptotics of the Airy function (140).

Now we check that the approximate expressions for the current densities (135) obey the (approximate) conservation law. The equality (95) is equivalent to

$$\begin{aligned} \sum_{n=1}^{\infty} e^{in\psi_0} \int_{-\pi}^{\pi} d\varphi \text{Ai}(B + a \cos \varphi - n\tilde{\Omega}) \cos n\varphi \\ \times \left[n\tilde{\Omega} - a \cos \varphi + \frac{\omega^2/2\Omega^2}{1 + \omega^2/2\Omega^2} (B + a \cos \varphi - n\tilde{\Omega}) \cos 2\varphi \right] = 0, \end{aligned} \tag{151}$$

where we have used the relation $\text{Ai}''(x) = x\text{Ai}(x)$. It is easy to verify that the terms on the left-hand side violating this equality are either proportional to a or $\text{Ai}(B - 2\tilde{\Omega})$ and so they are small. From (149) we see that such terms do not affect the leading contribution and appear in the power density as the small correction proportional to a^2 . In particular, had we replaced $\sin^2 \psi$ entering j_+ , which is presented in (135), by $1/2$, the expression standing in the square brackets in (149) would have changed: the coefficients at the first and second terms would have become 6 and 2, respectively. Therefore the expression for this correction is just an estimate.

5.2.2. Circularly polarized wave

Let us turn to the case of a circularly polarized plane electromagnetic wave of constant amplitude. The analysis is carried out in a complete analogy with the case of a linearly polarized electromagnetic wave. The asymptotics of the physical solutions to the LD equation was obtained by us in (43). In order to match the notation with the case of a linearly polarized wave, we denote $\varphi \rightarrow \psi$. Then, on integrating (43) and discarding the terms of the order of $O((\Omega x_-)^{-1})$ in comparison with the main contribution, we have

$$\begin{aligned} x_1 &\approx - \frac{\lambda \omega^3}{\Omega^2} x_- \cos \psi, & x_3 &\approx - \frac{\lambda \omega^3}{\Omega^2} x_- \sin \psi, \\ x_+ &\approx \frac{\lambda^2 \omega^4}{3} \left(1 + \frac{\omega^2}{\Omega^2} \right) x_-^3 & \psi &= \Omega x_- + \psi_0. \end{aligned} \tag{152}$$

After the replacement

$$x_- = y \left[\frac{\lambda^2 \omega^4}{2} \left(1 + \frac{\omega^2}{\Omega^2} \right) k_- \right]^{-1/3}, \tag{153}$$

we obtain

$$\begin{aligned}
 k_\mu x^\mu &= \frac{y^3}{3} + y \left(\frac{k_+}{2} + \frac{\lambda \omega^3 k_\perp}{\Omega^2} \cos \tilde{\psi} \right) \left[\frac{\lambda^2 \omega^4}{2} \left(1 + \frac{\omega^2}{\Omega^2} \right) k_- \right]^{-1/3} \\
 &=: \frac{y^3}{3} + By + ay \cos \tilde{\psi},
 \end{aligned}
 \tag{154}$$

where

$$\tilde{\psi} = \tilde{\Omega} y + \psi_0 + \bar{\psi}_0, \quad \cos \bar{\psi}_0 = \frac{k_1}{k_\perp}, \quad \tilde{\Omega} := \Omega \left[\frac{\lambda^2 \omega^4}{2} \left(1 + \frac{\omega^2}{\Omega^2} \right) k_- \right]^{-1/3},
 \tag{155}$$

and $k_\perp := \sqrt{k_1^2 + k_3^2}$. The Fourier transforms of the current density components take the form

$$\begin{aligned}
 j_- &= \left[\frac{\lambda^2 \omega^4}{2} \left(1 + \frac{\omega^2}{\Omega^2} \right) k_- \right]^{-1/3} \int_0^\infty dy e^{-ik_\mu x^\mu}, \quad j_+ = \frac{2}{k_-} \int_0^\infty dy y^2 e^{-ik_\mu x^\mu}, \\
 j_1 &= \frac{\lambda \omega^3}{\Omega} \left[\frac{\lambda^2 \omega^4}{2} \left(1 + \frac{\omega^2}{\Omega^2} \right) k_- \right]^{-2/3} \int_0^\infty dy y \sin \psi e^{-ik_\mu x^\mu}, \\
 j_3 &= -\frac{\lambda \omega^3}{\Omega} \left[\frac{\lambda^2 \omega^4}{2} \left(1 + \frac{\omega^2}{\Omega^2} \right) k_- \right]^{-2/3} \int_0^\infty dy y \cos \psi e^{-ik_\mu x^\mu}.
 \end{aligned}
 \tag{156}$$

Representing the integrand expressions as Fourier series and assuming that the estimate (142) is satisfied (for B and $\tilde{\Omega}$ defined in (154)), we deduce as in the case of a linearly polarized wave

$$\begin{aligned}
 \langle \text{Re}(j_-^* j_+) \rangle &\approx - \left(\frac{2}{k_-} \right)^{4/3} \left[\lambda^2 \omega^4 \left(1 + \frac{\omega^2}{\Omega^2} \right) \right]^{-1/3} \\
 &\times \sum_{n=1}^\infty \int_{-\pi}^\pi d\varphi' \text{Ai}(B + a \cos \varphi' - n\tilde{\Omega}) \cos n\varphi' \int_{-\pi}^\pi d\varphi \text{Ai}''(B + a \cos \varphi - n\tilde{\Omega}) \cos n\varphi.
 \end{aligned}
 \tag{157}$$

We see that the shift of the phase ψ by $\bar{\psi}_0$ is completely cancelled out. As for the components $|j_{1,3}|^2$, this is not so:

$$\begin{aligned}
 \langle |j_1|^2 \rangle &\approx \frac{\lambda^2 \omega^6}{4\Omega^2} \left[\frac{\lambda^2 \omega^4}{2} \left(1 + \frac{\omega^2}{\Omega^2} \right) k_- \right]^{-4/3} \sum_{n=1}^\infty \left| \int_{-\pi}^\pi d\varphi \text{Ai}'(B + a \cos \varphi - n\tilde{\Omega}) \right. \\
 &\quad \times \left. \left[\cos[(n+1)\varphi] e^{i\bar{\psi}_0} - \cos[(n-1)\varphi] e^{-i\bar{\psi}_0} \right] \right|^2, \\
 \langle |j_3|^2 \rangle &\approx \frac{\lambda^2 \omega^6}{4\Omega^2} \left[\frac{\lambda^2 \omega^4}{2} \left(1 + \frac{\omega^2}{\Omega^2} \right) k_- \right]^{-4/3} \sum_{n=1}^\infty \left| \int_{-\pi}^\pi d\varphi \text{Ai}'(B + a \cos \varphi - n\tilde{\Omega}) \right. \\
 &\quad \times \left. \left[\cos[(n+1)\varphi] e^{i\bar{\psi}_0} + \cos[(n-1)\varphi] e^{-i\bar{\psi}_0} \right] \right|^2.
 \end{aligned}
 \tag{158}$$

Substituting the expressions (157) and (158) into (78), we obtain the spectral density of radiation formed on the asymptotics of the physical solutions to the LD equation in the field of a circularly polarized plane wave. Notice that this expression is already independent of the phase $\bar{\psi}_0$ and represents a function of k_0 and k_\perp only. This property is quite expectable from the symmetry arguments for the case of a circularly polarized incident electromagnetic wave.

When a is small and the estimation (150) is fulfilled, expanding the expressions obtained in a Taylor series in a and keeping only the leading contribution, we derive

$$\begin{aligned}
 \langle \text{Re}(j_{-j_+}^*) \rangle &\approx -(2\pi)^2 \left(\frac{2}{k_-}\right)^{4/3} \left[\lambda^2 \omega^4 \left(1 + \frac{\omega^2}{\Omega^2}\right) \right]^{-1/3} \left(\frac{a}{2}\right)^2 \text{Ai}'(B - \tilde{\Omega}) \text{Ai}'''(B - \tilde{\Omega}), \\
 \langle |j_1|^2 \rangle &\approx (2\pi)^2 \frac{\lambda^2 \omega^6}{4\Omega^2} \left[\frac{\lambda^2 \omega^4}{2} \left(1 + \frac{\omega^2}{\Omega^2}\right) k_- \right]^{-4/3} \left\{ [\text{Ai}'(B - \tilde{\Omega})]^2 + \left(\frac{a}{2}\right)^2 \right. \\
 &\quad \left. \times \left[(2 - \cos 2\tilde{\psi}_0) \text{Ai}'(B - \tilde{\Omega}) \text{Ai}'''(B - \tilde{\Omega}) + [\text{Ai}''(B - 2\tilde{\Omega})]^2 \right] \right\}, \\
 \langle |j_3|^2 \rangle &\approx (2\pi)^2 \frac{\lambda^2 \omega^6}{4\Omega^2} \left[\frac{\lambda^2 \omega^4}{2} \left(1 + \frac{\omega^2}{\Omega^2}\right) k_- \right]^{-4/3} \left\{ [\text{Ai}'(B - \tilde{\Omega})]^2 + \left(\frac{a}{2}\right)^2 \right. \\
 &\quad \left. \times \left[(2 + \cos 2\tilde{\psi}_0) \text{Ai}'(B - \tilde{\Omega}) \text{Ai}'''(B - \tilde{\Omega}) + [\text{Ai}''(B - 2\tilde{\Omega})]^2 \right] \right\}.
 \end{aligned} \tag{159}$$

As a result, taking into account that $\omega/\Omega \gg 1$, we come to

$$\begin{aligned}
 \langle \text{Re}(j_{-j_+}^*) - |j_1|^2 - |j_3|^2 \rangle &\approx -\left(\frac{2\pi}{\omega}\right)^2 \left(\frac{\sqrt{2}\Omega}{\lambda k_-}\right)^{2/3} \left\{ [\text{Ai}'(B - \tilde{\Omega})]^2 + \left(\frac{a}{2}\right)^2 \right. \\
 &\quad \left. \times \left[4\text{Ai}'(B - \tilde{\Omega}) \text{Ai}'''(B - \tilde{\Omega}) + [\text{Ai}''(B - 2\tilde{\Omega})]^2 \right] \right\}.
 \end{aligned} \tag{160}$$

Notice that the approximate expressions for the current density leading to (157) and (158) obey the charge conservation law up to the terms of the order of a .

In the leading order ($a = 0$), the spectral density of radiation (formed on the asymptotics) taken at a given fixed photon energy k_0 is an oscillating function of k_\perp both for the circularly (160) and linearly polarized (149) incident electromagnetic waves. The minima of the intensity of radiation are specified by the zeros of the function $\text{Ai}'(x)$:

$$B - \tilde{\Omega} \approx -1.02; -3.25; -4.82 \dots \tag{161}$$

For the fixed energy k_0 of photons, the radiation intensity formed on the asymptotics when projected to the plane normal to the direction of propagation of the electromagnetic wave (in our case the x^2 axis) will look like the system of concentric rings. In the case of a linearly polarized laser beam, the small corrections proportional to a deform this pattern, but qualitatively it remains the same. Since the maxima of the radiation intensity are observed at the angles which depend on the energy of the emitted photon, the broadband detector will see this radiation as a circular rainbow.

6. Conclusions

Concluding the investigation, we may sum up the main results we have achieved. First, we ascertained once again that the LD equation provides a solid basis for the description of the effective dynamics of charged particles. The use of it allowed us to derive the late time asymptotics of particle's motion and to establish some general properties of this motion, which are in accordance with our expectations based on a physical intuition. Nothing pathological happens during the evolution obeying the LD equation, when the physical solutions are only taken into account. Second, we completely described the asymptotic regimes of motion of charged particles in the electromagnetic fields of simple configurations and revealed some peculiar properties of the motion in these regimes. One of these properties is a striking feature of the total radiation power, which, at the large proper-times, turns out to be independent of the charge and the external field strength for a charged particle moving in the electromagnetic plane wave of constant amplitude. This clearly shows a nonperturbative character

(from the viewpoint of quantum electrodynamics) of the late time asymptotics obtained. Third, employing the asymptotics found, we obtained the spectral density of radiation formed on these asymptotics and established the specific features of such a radiation so that it can be firmly identified in an experiment. This radiation is produced by de-excited electrons and is predominantly constituted by soft photons. That is why we may expect that the classical theory of radiation is reliable in this regime. Of course, it would be interesting to take the quantum corrections into account or to describe the polarization properties of this radiation, but we leave these topics for a future research.

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