Asymptotics of physical solutions to the Lorentz-Dirac equation for planar motion in constant electromagnetic fields

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We present a study of planar physical solutions to the Lorentz-Dirac equation in a constant electromagnetic field. In this case, we reduced the Lorentz-Dirac equation to one second-order differential equation. We obtained the asymptotics of physical solutions to this equation at large proper times. It turns out that, in a crossed constant uniform electromagnetic field with vanishing invariants, a charged particle enters a universal regime at large times. We found that the ratios of momentum components that tend to constants are determined only by the external field. This effect is essentially due to a radiation reaction. There is no such effect for the Lorentz equation in this field.

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I. INTRODUCTION

The Lorentz-Dirac (LD) equation suffers from various types of inconsistencies. The latter come from the higher-derivative Schott term entering the LD equation and appearing as blowing up (runaway) and acausal solutions. However, despite its undesirable features, we have to accept the LD equation as correct in its range of applicability for the following reasons. First, as shown in the seminal paper by Dirac [1] and then more elaborately in [2], the LD equation stems from the energy-momentum conservation law, provided a charged particle is sufficiently small and possesses negligible higher multipoles of the charge distribution. Second, under these assumptions the LD equation is a minimal evolutionary equation describing a radiation reaction which complies with all the symmetries of the model: Poincaré and reparametrization invariance. Furthermore, after certain approximations, the LD equation was derived in the context of quantum electrodynamics (see, e.g., [3]), where it can be considered as a leading quasiclassical asymptotic to the Schwinger-Dyson equations for an electron. Therefore, the LD equation makes physical sense and, under certain conditions, its solutions should give rise to predictions which can be observed in experiments. It is clear that the LD equation is valid in the range of energies and field strengths where the quantum corrections are negligible in comparison with the classical contribution. Rough general estimates of this range can be found, e.g., in [4,5], and a more accurate analysis for the particular case of a constant homogeneous magnetic field is presented in [6]. Various generalizations of the LD equation to include spin and higher multipoles [7] or an interaction with non-Abelian gauge fields [8] and gravity [9], to higher dimensions [10], and to dyonic [11] and massless charged particles [12] are also known. All of them have higher-derivative terms and, hence, possess the same unwanted properties as the LD equation.

There is a coherent approach [4,13–18] for extracting physical information from the LD equation and its analogs. It is based on the notion of a physical solution. In a general setting, it is as follows. Given a system of interacting fields ϕ_a^γ with the action functional

$$S[\phi_1^\gamma,\dots,\phi_N^\gamma]=\sum_{a=1}^{N}\lambda_a^{-1}S_a^0[\phi_a^\gamma]+S_{int}[\phi_1^\gamma,\dots,\phi_N^\gamma], \quad (1)$$

where γ is a condensed index representing the group and spacetime indices and spacetime points, a enumerates the fields, and λ_a are some constants. Then the solution $\phi_a^\gamma(\lambda)$ of the coupled system of equations of motion corresponding to the action (1) at given initial and boundary conditions is called physical if there exist finite limits

$$\lim_{\lambda_b\to 0}\phi_a^\gamma(\lambda), \quad a=\overline{1,N}, \quad (2)$$

the other λ's being fixed. This regularity condition completely rules out runaway solutions to the LD equation. In addition, its physical solutions are unambiguously specified by the six initial data—three position coordinates and three momentum components—as they should be in the realm of Newtonian mechanics.

Since the LD equation is nonlinear, it is hard to solve it even in simple external field configurations. Almost all the exact solutions to the LD equation can be found in [14,15,17,19,20]. In this paper, we address the problem of finding and describing exact physical solutions to the LD equation in a constant homogeneous electromagnetic field. Moreover, we restrict ourself to only planar motion. Even in this rather simple situation we did not succeed in finding exact essentially planar (i.e., nonlinear) solutions to the LD equation. However, we reduce the LD equation to one second-order differential equation and investigate the asymptotics of its physical solutions at large times. For a constant homogeneous magnetic field this asymptotics was found in [15]. As far as constant electric and crossed fields are concerned these asymptotics, to our knowledge, are obtained for the first time. It turns out that, in the crossed field configuration, the LD equation possesses an attractor, and the system passes into a universal regime at large times. After a lapse of time, the identical charged particles moving on the plane in such an electromagnetic field "forget" their initial data. Their trajectories become parallel and certain ratios of momentum

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components tend to constants that are independent of the initial conditions and determined only by the external field. This effect is essentially due to the radiation reaction. It is absent for the solution to the Lorentz equation in this field and can serve as an explicit manifestation of the validity of classical radiation reaction theory in the domain of its applicability.

The paper is organized as follows. In Sec. II, we present general formulas for the radiation reaction and define the physical solutions to the LD equation. Here we also give an integro-differential equation for the physical solutions. Section III is the main part of the paper. In Sec. III A, we briefly describe linear motion and exact solutions to the LD equation in this case. In Sec. III B 1, we investigate the symmetries of the LD equation and provide the necessary and sufficient condition for the motion of a charged particle to be planar. In Sec. III B 2, we derive the second-order differential equation describing planar solutions to the LD equation. Section III B 3 is devoted to the asymptotics of the physical solutions to the LD equation at large times. In Sec. III B 4, we consider the same problem in the framework of the so-called Landau-Lifshitz equation [21]. Its solutions in external fields of such configurations are known: for a constant homogeneous magnetic field, see, e.g., [14,22], and for crossed fields with vanishing invariants, see [23]. In Sec. III B 5, we investigate the stability of the obtained asymptotics to the solutions of the exact LD equation with crossed fields against external electromagnetic field fluctuations. A rough constraint on these fluctuations is derived. In conclusion, we summarize the main results of the paper and discuss the prospects for further research.

II. GENERAL FORMULAS

Consider a particle with charge e and bare mass $\bar{m}$ interacting with the electromagnetic field A_μ on the Minkowski background $\mathbb{R}^{1,3}$ with the metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and coordinates x^μ, $\mu = \overline{0,3}$. The action functional for such a system has the form

$$S[x(\tau),A(x)] = -\bar{m}\int d\tau\sqrt{\dot{x}^2} - e\int d\tau A_\mu \dot{x}^\mu - \frac{1}{16\pi}\int d^4x F_{\mu\nu}F^{\mu\nu}, \tag{3}$$

where $x^\mu(\tau)$ defines the particle worldline, $F_{\mu\nu} := \partial_{[\mu}A_{\nu]}$ is the strength tensor of the electromagnetic field (the square brackets denote antisymmetrization without 1/2)

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{bmatrix}, \tag{4}$$

and we take the system of units in which the speed of light $c = 1$. In the proper time parametrization $\dot{x}^2 = 1$, the LD equation [1,24] reads

$$m\ddot{x}_\mu = eF_{\mu\nu}\dot{x}^\nu + \frac{2}{3}e^2(\dddot{x}_\mu + \ddot{x}^2\dot{x}_\mu), \tag{5}$$

where m is a physical (renormalized) mass and $F_{\mu\nu}$ is the strength tensor of the external electromagnetic field. Introducing the dimensionless quantities

$$x^\mu \to m^{-1}x^\mu, \quad \tau \to m^{-1}\tau, \quad F_{\mu\nu} \to m^2e^{-1}F_{\mu\nu}, \tag{6}$$

we rewrite it in the form

$$\dot{\upsilon}_\mu = f_\mu + \lambda(\ddot{\upsilon}_\mu + \dot{\upsilon}^2\upsilon_\mu), \quad f_\mu := F_{\mu\nu}\upsilon^\nu, \tag{7}$$

where $\lambda = 2e^2/3$ and $m\upsilon^\mu = m\dot{x}^\mu$ is the four-momentum of the particle.

The LD equation possesses unphysical solutions. Following [13,14], we shall call the solution $x^\mu(\lambda,\tau)$ physical if it tends to the solution $x^\mu(0,\tau)$ of the corresponding Lorentz equation as λ goes to zero. This is a realization of the general definition given in the Introduction in the case of classical electrodynamics. According to this definition, the physical solution should be regular at large mass m and small e^2. From Eq. (5) we see that this requirement leads to a regularity of the physical solution with respect to the external field and the parameter λ entering (7). The former simply follows from the general theorems regarding the dependence of solutions to ordinary differential equations on a parameter (see, e.g., [25]), while the latter condition is not trivial. Also notice that all the known physically reasonable solutions to the LD equation are physical in the sense adopted by us. Some extra arguments in favor of this definition of physical solutions are given in Appendix.

We can find these solutions perturbatively as a (formal) series in λ. This perturbative scheme reduces the order of the LD equation and provides a unique solution to it at some fixed initial position and velocity of the particle. The first iteration of the perturbative procedure yields the Landau-Lifshitz equation [21]. It is not difficult to write an integro-differential equation that describes the physical solutions to the LD equation [4,15–17]. In the proper time parametrization, it is

$$\dot{\upsilon}_\mu(\tau) = \mathrm{pr}_\mu^\nu(\tau)\int_0^\infty dt e^{-t}\mathcal{P}_\nu(\tau+\lambda t), \quad \mathrm{pr}_\mu^\nu := \delta_\mu^\nu - \upsilon_\mu\upsilon^\nu, \tag{8}$$

where $\mathcal{P}_\mu = f_\mu + \lambda\dot{\upsilon}^2\upsilon_\mu$. Solutions to Eq. (8) are solutions to the LD equation (7) with $\upsilon_\mu\dot{\upsilon}^\mu = 0$. It is the latter requirement that gives rise to the projector entering Eq. (8). The solutions of this equation tend to the solutions of the Lorentz equation at $\lambda \to 0$. Expanding Eq. (8) in a series in λ, we see that solutions to Eq. (8) are those solutions to the LD equation which are obtained from it by the aforementioned perturbative scheme (for details, see, e.g., [4]). If we knew all the terms of the perturbation series for the four-acceleration $\dot{\upsilon}^\mu(\tau,\lambda)$ then formula (8) would tell us that this series in λ must be summed by the Borel method [26]. So, if the following conditions are satisfied at some fixed initial position and velocity, then a unique solution to Eq. (8) exists at sufficiently small $\lambda = \lambda_0 > 0$:

(1) There exists a unique solution to the corresponding Lorentz equation, which is defined at any $\tau > \tau_0$ and tends to infinity not faster than $Me^{a\tau}$ for $\tau \to \infty$. Here, τ_0, M, and $a > 0$ are some constants.

(2) The perturbative series in λ converges absolutely in a vicinity of the point $\lambda = 0$ at sufficiently small λ.

If the value of λ_0 is smaller than the physical value of the parameter λ then the physical solution to the LD equation at the physical value of λ is obtained by an analytical continuation in λ.

A concrete prescription for construction of this analytical continuation depends on analytical properties of the solution to the Lorentz equation. For example, if this solution satisfies the first condition above and has a finite number of singularities in the part of the complex τ plane where $\mathrm{Re}\tau > \tau_0$ and $\mathrm{Im}\tau > 0$ or $\mathrm{Re}\tau > \tau_0$ and $\mathrm{Im}\tau < 0$ for some τ_0, then the physical solution $x^\mu(\lambda,\tau)$ to the LD equation has the same properties at sufficiently small λ. In that case, we can rotate the ray along which the integration contour tends to infinity so as to make the integral (8) convergent for any a. In particular, this procedure makes the right-hand side of Eq. (8) convergent when we perturbatively solve Eq. (8) by Picard iterations starting with the solution to the Lorentz equation $x^\mu(0,\tau)$ satisfying the first condition above, while $3a\lambda \geqslant 1$. Of course, the latter situation is rather unphysical since for the solutions to the Lorentz equation

$$a\lambda \sim \frac{2\alpha}{3}\frac{E}{E_0}, \quad E_0 = \frac{m^2}{|e|\hbar} \approx 4.41 \times 10^{13}\ \mathrm{G}, \tag{9}$$

where E is a characteristic value of the field strength, $\alpha \approx 1/137$ is the fine structure constant, and E_0 is the Schwinger field. However, we can define the physical solution even in this case.

Another, possibly more convenient, form of Eq. (8) can be derived if we write [15]

$$\dot\upsilon^2(\tau) = \int_0^\infty dt e^{-t}(\dot\upsilon f)(\tau + \lambda t/2) \tag{10}$$

for physical solutions. Then

$$\dot\upsilon_\mu(\tau) = \mathrm{pr}^\nu_\mu(\tau)\int_0^\infty dt e^{-t}\bigg[f_\nu(\tau+\lambda t) + \lambda\int_0^t ds(\dot\upsilon f) \times(\tau+\lambda s/2)\upsilon_\nu\left[\tau+\lambda(t-s)\right]\bigg]. \tag{11}$$

In this form, it is obvious that physical solutions to the LD equation are straight lines in the spacetime in the absence of external fields. If the charged particle leaves a region with a nonzero electromagnetic field it will move uniformly along a straight line in the future.

The integro-differential equations for physical solutions to the LD equation, which we have presented in this section, are not very useful in finding analytical solutions. But they are pertinent to numerical simulations, for example, by the Picard iterations, rather than for the LD equation itself. Even if we set the initial conditions to their physical values and solve the LD equation numerically, we shall obtain an unphysical runaway solution owing to machine approximation errors. Also, Eqs. (8) and (11) are a good starting point for an investigation of the stochastic LD equation (see, e.g., [3,27]). As in the case of numerical simulations, the unphysical solutions should be explicitly excluded to get rid of stochastically induced runaways. The study of this problem will be given elsewhere.

III. PHYSICAL SOLUTIONS FOR A PLANAR MOTION

In this section, we study the planar motion of a charged particle obeying the LD equation. We call the motion of a particle planar (linear) provided that the trajectory of this particle in the space can be made planar (linear) by an appropriate Lorentz transform. For linear motion, the worldline of a particle lies in a two-dimensional plane of the spacetime. For planar motion, it lies in a three-dimensional hyperplane. Throughout this section, we mostly assume that the particle moves in a constant homogeneous external electromagnetic field, but some results can be generalized to a nonconstant electromagnetic field of a special configuration. This will be mentioned in its place.

A. Linear motion

Let us consider first hyperbolic motion, that is, motion with vanishing LD force,

$$\begin{aligned}\ddot\upsilon_\mu + \dot\upsilon^2\upsilon_\mu = 0 &\Rightarrow \dot\upsilon^2 = -\omega^2 = \mathrm{const}\\ &\Rightarrow \upsilon_\mu(\tau) = \alpha_\mu\cosh(\omega\tau) + \beta_\mu\sinh(\omega\tau),\\ &\quad\alpha^2 = -\beta^2 = 1, \quad \alpha_\mu\beta^\mu = 0.\end{aligned} \tag{12}$$

The hyperbolic motion is the solution to the LD equation (7) with a constant and homogeneous external electromagnetic field if, and only if,

$$\ddot\upsilon^\mu = F^\mu_{\ \rho}F^\rho_{\ \nu}\upsilon^\nu = -\omega^2\upsilon^\mu. \tag{13}$$

By the use of canonical forms (see [28] and also below) of the strength tensor $F_{\mu\nu}$, it is not difficult to show that the last equality (the equation on eigenvectors) is fulfilled if, and only if, there exists a Lorentz frame in which the particle moves along three-vectors of the electric and magnetic field strengths. In this system of coordinates, the problem reduces to a description of linear motion.

For linear motion, the LD equation (7) is equivalent to (see, e.g., [1,15,20])

$$\begin{aligned}\frac{\dot\upsilon}{\sqrt{1+\upsilon^2}} &= \lambda\frac{d}{d\tau}\frac{\dot\upsilon}{\sqrt{1+\upsilon^2}} + E\\ &\Rightarrow \upsilon(\tau) = \sinh(c_2 + \upsilon\tau + c_1 e^{\lambda^{-1}\tau}),\end{aligned} \tag{14}$$

where $\upsilon(\tau) = \dot x(\tau)$. A generalization of this solution to the case where the electric field depends on τ is trivial. The solution (14) with $E = 0$ is also a general solution to the free LD equation written in the Lorentz frame, where the initial three-velocity and three-acceleration are parallel. The solution (14) becomes physical if we take $c_1 = 0$. It is easy to verify that the solution (14) satisfies the integro-differential equation (8) only at vanishing c_1. In order to make the integral convergent at $\lambda E \geqslant 1$, we have to rotate the integration contour in Eq. (8) as described in the previous section.

B. Planar motion

1. General considerations

Let us turn to planar motion. When one of the Poincaré invariants of the electromagnetic field is not zero, the strength tensor (4) can be represented as

$$F^{\mu\nu} = \omega_1 e_0^{[\mu}e_1^{\nu]} + \omega_2 e_2^{[\mu}e_3^{\nu]}, \quad (e_\alpha e_\beta) = \eta_{\alpha\beta}, \tag{15}$$

where e^μ_α, $\alpha = \overline{0,3}$, is a tetrad of eigenvectors of the tensor $(F^2)^\mu_\nu$. The eigenvalue ω_1^2 of the tensor $(F^2)^\mu_\nu$ corresponds to the vectors $e^\mu_{0,1}$, and the eigenvalue $-\omega_2^2$ corresponds to the vectors $e^\mu_{2,3}$. In terms of the Poincaré invariants of the electromagnetic field $I_1 = \mathbf{E}^2 - \mathbf{H}^2$ and $I_2 = 2(\mathbf{EH})$, these ω's read as

$$\omega_1^2 = (\sqrt{I_1^2 + I_2^2} + I_1)/2, \quad \omega_2^2 = (\sqrt{I_1^2 + I_2^2} - I_1)/2. \quad (16)$$

The Lorentz transforms, which do not change the strength tensor (15), constitute the group SO(1,1) × SO(2). It has the matrix representation

$$\Lambda^{\mu\nu} = \left(e_0^\mu e_0^\nu + e_1^\mu e_1^\nu\right)\cosh\psi + e_0^{(\mu} e_1^{\nu)}\sinh\psi + \left(e_2^\mu e_2^\nu + e_3^\mu e_3^\nu\right) \times \cos\varphi + e_2^{[\mu} e_3^{\nu]}\sin\varphi, \quad (17)$$

where ψ and φ are the group parameters and the parentheses on the indices mean a symmetrization without 1/2.

In the degenerate case, when $I_1 = I_2 = 0$, the strength tensor is given by

$$F^{\mu\nu} = \omega e_-^{[\mu} e_1^{\nu]}, \quad (e_a e_b) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad (18)$$

where e^μ_a, $a = \{-,1\}$, are eigenvectors of the tensor $(F^2)^\mu_\nu$ corresponding to zero eigenvalue. The normalized eigenvector e^μ_1 is orthogonal to the vector e^μ_3, which, in turn, is a normalized eigenvector of the tensors F^μ_ν and $(F^2)^\mu_\nu$ corresponding to zero eigenvalue. These conditions determine the vectors e_1 and e_3 uniquely up to addition of the isotropic vector e_- and inversion. The factor ω can be included in e_-, but we leave it in the expression so as to control the external field. The strength tensor (18) is invariant with respect to the two-dimensional Abelian subgroup of the Lorentz group generated by the two elements

$$\Lambda_1^{\mu\nu} = \eta^{\mu\nu} + r_1 e_3^{[\mu} e_-^{\nu]} + \frac{r_1^2}{2} e_-^\mu e_-^\nu,$$
$$\Lambda_2^{\mu\nu} = \eta^{\mu\nu} + r_2 e_1^{[\mu} e_-^{\nu]} + \frac{r_2^2}{2} e_-^\mu e_-^\nu, \quad (19)$$

where r_1 and r_2 are the group parameters. Thus, in both the degenerate and nondegenerate cases, we anticipate two integrals of motion of the LD equation provided the eigenvectors of the tensors F^μ_ν and $(F^2)^\mu_\nu$ do not depend on the point in spacetime.

In order to obtain these integrals of motion, it is useful to introduce new variables adjusted to the action of the symmetry group. In the nondegenerate case, they are

$$\mathrm{v}_0 = \sqrt{u_e}\cosh\psi, \quad \mathrm{v}_1 = \sqrt{u_e}\sinh\psi,$$
$$\mathrm{v}_2 = \sqrt{u_h}\sin\varphi, \quad \mathrm{v}_3 = \sqrt{u_h}\cos\varphi, \quad (20)$$

where $\mathrm{v}_\alpha := (e_\alpha \upsilon)$ are the projections of the momentum on the tetrad vectors, $\varphi(\tau)$ and $\psi(\tau)$ are the symmetry group parameters, and $u_e(\tau) - u_h(\tau) = 1$. Then, convolving the LD equation (7) with the eigenvectors and combining the equations obtained, we have

$$u_e(\tau)\dot\psi(\tau) = c_1 e^{\lambda^{-1}\tau} + \int_\tau^\infty \frac{dt}{\lambda} e^{-\lambda^{-1}(t-\tau)}\omega_1 u_e(t),$$
$$u_h(\tau)\dot\varphi(\tau) = c_2 e^{\lambda^{-1}\tau} - \int_\tau^\infty \frac{dt}{\lambda} e^{-\lambda^{-1}(t-\tau)}\omega_2 u_h(t). \quad (21)$$

In the degenerate case, the analogous variables read as

$$\mathrm{v}_1 = \mathrm{v}_- r_2, \quad \mathrm{v}_3 = \mathrm{v}_- r_1, \quad \mathrm{v}_+ = \mathrm{v}_-^{-1} + \mathrm{v}_-\left(r_1^2 + r_2^2\right), \quad (22)$$

where e^μ_+ is an isotropic vector orthogonal to e^μ_1 and e^μ_3, and such that $(e_+ e_-) = 2$, the momentum projections are defined as $\mathrm{v}_\alpha := (e_\alpha \upsilon)$, $\alpha = \{\pm,1,3\}$, and the functions $r_1(\tau)$ and $r_2(\tau)$ are the symmetry group parameters. The respective integrals of motion become

$$u(\tau)\dot r_1(\tau) = c_1 e^{\lambda^{-1}\tau},$$
$$u(\tau)\dot r_2(\tau) = c_2 e^{\lambda^{-1}\tau} + \int_\tau^\infty \frac{dt}{\lambda} e^{-\lambda^{-1}(t-\tau)}\omega u(t), \quad (23)$$

where $u(\tau) := \mathrm{v}_-^2(\tau)$. The nonvanishing constants c_1 and c_2 in Eqs. (21) and (23) correspond to unphysical solutions. So they should be set to zero and we do not take them into account henceforth.

Consider the particular case $I_2 = 0$ (see [14]). If the invariant $I_1 > 0$ then ω_2 is zero and we get from Eqs. (20) and (21)

$$\mathrm{v}_2 = \mathrm{v}_3 = 0 \quad \text{or} \quad \dot\varphi = 0. \quad (24)$$

The first case represents a linear motion, the second case a planar one. When the invariant $I_1 < 0$, we have $\omega_1 = 0$ and

$$\dot\psi = 0. \quad (25)$$

In the degenerate case $I_1 = I_2 = 0$, we obtain from Eq. (23)

$$\dot r_1 = 0. \quad (26)$$

By definition of the variables ψ and r_1 as the group parameters, the particle can be confined to a plane by an appropriate Lorentz transform in these cases [see Eqs. (20) and (22) with vanishing parameters ψ and r_1].

Thus we have proved the following statement. If $I_2 = 0$ and the field strength tensor admits the representation (15) or (18) with the constant eigenvectors e^μ_α, then a charged particle obeying the LD equation executes a planar motion. In a constant homogeneous electromagnetic field, the converse statement is also true, namely, if a charged particle obeying the LD equation executes an essentially planar (i.e., nonlinear) motion then the invariant $I_2 = 0$.

2. *Second-order equation*

Now we investigate the planar motion in detail. Let us characterize this motion by the tetrad

$$e^\mu_\alpha \eta_{\mu\nu} e^\nu_\beta = \eta_{\alpha\beta}, \quad e^\mu_3 e^\alpha_\mu = 0, \quad (e_3)^2 = -1, \quad (27)$$

where the indices α and β have the values 0,1,2. They are raised and lowered by the metric $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1)$. The worldline of the particle and the external electromagnetic field admit the representation

$$\upsilon^\mu(\tau) = \upsilon^\alpha(\tau) e^\mu_\alpha, \quad e^\mu_3 \upsilon_\mu(\tau) = 0,$$
$$F_{\mu\nu} = f_{\alpha\beta} e^\alpha_\mu e^\beta_\nu, \quad f_{\alpha\beta} = \omega\varepsilon_{\alpha\gamma\beta}\xi^\gamma, \quad (28)$$

where $\varepsilon_{012} = 1$ and $\xi^2 = \{\pm1,0\}$. The LD equation (7) is rewritten as (for the Lorentz equation see, e.g., [29])

$$\dot\upsilon_\alpha = \omega\varepsilon_{\alpha\beta\gamma}\xi^\beta\upsilon^\gamma + \lambda(\ddot\upsilon_\alpha + \dot\upsilon^2\upsilon_\alpha), \quad \upsilon^2 = 1. \quad (29)$$

We see that the LD equation of a charged particle confined to a plane possesses a symmetry. This is a residue of the symmetry discussed above after reduction to a plane. The residual symmetry group is constituted by the Lorentz transforms leaving the vector ξ^α intact. So, if $\xi^2 \leqslant 0$, this symmetry group is isomorphic to SO(1,1), and if $\xi^2 > 0$ it is isomorphic to SO(2). This symmetry allows us to reduce the problem of integration of the system of equations (29) to an integration of an autonomous system of three first-order equations or one second-order equation.

To this end, we introduce new more convenient variables

$$m^\alpha := \varepsilon^{\alpha\beta\gamma}\dot{\upsilon}_\beta\upsilon_\gamma. \tag{30}$$

In these variables, the LD equation (29) turns into a system of first-order equations:

$$\begin{gathered}\lambda\dot{m}_\alpha = m_\alpha + \omega(\xi_\alpha - p\upsilon_\alpha), \quad \dot{\upsilon}_\alpha = -\varepsilon_{\alpha\beta\gamma}m^\beta\upsilon^\gamma,\\ m_\alpha\upsilon^\alpha = 0, \quad \upsilon^2 = 1,\end{gathered} \tag{31}$$

where $p := \xi_\alpha\upsilon^\alpha$. We see that the vector field of this system of first-order differential equations depends analytically on the vector ξ_α. Therefore the solutions to this system depend analytically on ξ_α too and turn smoothly one into another when one changes the vector ξ_α. In particular, this holds when we smoothly change the square ξ^2 from positive to negative values.

Then we introduce the invariants of the symmetry group action:

$$\begin{gathered}a = \xi_\alpha m^\alpha, \quad b = p^{-1}\varepsilon^{\alpha\beta\gamma}\xi_\alpha m_\beta\upsilon_\gamma = -p^{-1}\dot{p},\\ s = p^{-2}(m^2 - \xi^2a^2), \quad u = p^2 - \xi^2.\end{gathered} \tag{32}$$

These invariants are dependent. From their definition, it is not difficult to obtain the identity

$$a^2\frac{1+\xi^2u}{\xi^2+u} + b^2 = -su. \tag{33}$$

The invariants (32) evolve according to the equations that follow from Eqs. (31):

$$\begin{gathered}\lambda\dot{a} = a - \omega u, \quad \dot{u} = -2b(\xi^2+u),\\ \lambda\dot{s} = 2s(1+\lambda b) + 2\omega a\frac{1+\xi^2u}{\xi^2+u},\\ \lambda\dot{b} = b + \lambda\xi^2s + \lambda a^2\frac{|\xi^2|-1}{u} = b - \lambda\frac{a^2+\xi^2b^2}{u}.\end{gathered} \tag{34}$$

The first equation in this system is one of the equations of the system (21) or the second equation in (23) written in a differential form. The physical solutions are described by a single integro-differential equation on the function $u(\tau)$. It is obtained from the above equations if we write

$$\begin{gathered}a(\tau) = \int_\tau^\infty \frac{dt}{\lambda}e^{-\lambda^{-1}(t-\tau)}\omega u(t),\\ p^2(\tau)s(\tau) = \int_\tau^\infty \frac{2dt}{\lambda}e^{-2\lambda^{-1}(t-\tau)}[1+\xi^2u(t)]\omega a(t)\end{gathered} \tag{35}$$

and substitute these expressions into the identity (33) with the function $b(\tau)$ taken from the second equation in the system (34). The solution of this integro-differential equation is specified by only one arbitrary constant. If this solution or some unphysical solution to the system (34) are known, we can integrate the LD equation.

Indeed, in general the vector $\dot{\upsilon}_\alpha$ can be expressed as a linear combination of the vectors ξ_α, υ_α, and $\varepsilon_{\alpha\beta\gamma}\xi^\beta\upsilon^\gamma$. The coefficients of this decomposition are certain functions of the invariants which are already known. Therefore, we need to integrate a system of linear equations with variable coefficients. So,

$$\dot{\upsilon}_\alpha = \frac{a}{u}\varepsilon_{\alpha\beta\gamma}\xi^\beta\upsilon^\gamma + \frac{bp}{u}(\xi_\alpha - p\upsilon_\alpha). \tag{36}$$

Because of the orthogonality condition, only two equations are independent. Now we make a substitution in Eq. (36) of the form (20),(22). In the case $\xi^\alpha = (1,0,0)$, we have

$$\upsilon_0 = p, \quad \upsilon_1 = \sqrt{u}\cos\varphi, \quad \upsilon_2 = \sqrt{u}\sin\varphi, \tag{37}$$

where $u = p^2 - 1$. Then we write the first and second components of Eq. (36) as

$$\begin{gathered}\frac{\dot{u}}{2\sqrt{u}}\cos\varphi - \sqrt{u}\sin\varphi\dot{\varphi} = \frac{a}{\sqrt{u}}\sin\varphi - \frac{bp^2}{\sqrt{u}}\cos\varphi,\\ \frac{\dot{u}}{2\sqrt{u}}\sin\varphi + \sqrt{u}\cos\varphi\dot{\varphi} = -\frac{a}{\sqrt{u}}\cos\varphi - \frac{bp^2}{\sqrt{u}}\sin\varphi,\end{gathered} \tag{38}$$

whence

$$u\dot{\varphi} = -a, \tag{39}$$

and the momentum υ_α is found by quadrature. If $\xi^\alpha = (0,0,1)$, we substitute

$$\upsilon_0 = \sqrt{u}\cosh\psi, \quad \upsilon_1 = \sqrt{u}\sinh\psi, \quad \upsilon_2 = p, \tag{40}$$

where $u = p^2 + 1$. A combination of the zeroth and first components of Eq. (36) results in

$$u\dot{\psi} = a. \tag{41}$$

In the third case $\xi^\alpha = (1,0,1)$, we have

$$\upsilon_1 = rp, \quad \upsilon_- = p, \quad p\upsilon_+ = 1 + ur^2, \tag{42}$$

where $\upsilon_+ := \upsilon_0 - \upsilon_2$, $\upsilon_- = \upsilon_0 + \upsilon_2$, and $u = p^2$. Then we arrive at

$$u\dot{r} = a \tag{43}$$

for the first component of Eq. (36). The case of an arbitrary vector ξ^α is reduced to the ones considered by a proper Lorentz transform of the tetrad indices. We should emphasize that Eqs. (39), (41), and (43) follow from Eq. (36) for purely kinematic reasons. We did not use the LD equation to obtain them. Notice also that the equations of motion (34), (39), (41), and (43) are valid for a nonconstant external field parameter $\omega(\tau,\xi_\alpha x^\alpha(\tau))$. In accordance with our general considerations, physical solutions are specified by four constants—two constants specify the initial position on the plane, one determines the function $u(\tau)$, and another one is needed to pick out the unique solution from Eqs. (39), (41), or (43).

Thus, we have to find the evolution of invariants described by Eqs. (34). In the case of a constant external field parameter ω, the autonomous system (34) is equivalent to one second-order differential equation in the function $a(u)$ or its inverse $u(a)$,

$$\begin{gathered}a'' = -\frac{[2au + \xi^2(a+\omega u)]a'}{2u(a-\omega u)(\xi^2+u)} - \frac{2\lambda^2a^2(\xi^2+u)a'^3}{u(a-\omega u)^2},\\ \ddot{u} = \frac{[2au + \xi^2(a+\omega u)]\dot{u}^2}{2u(a-\omega u)(\xi^2+u)} + \frac{2\lambda^2a^2(\xi^2+u)}{u(a-\omega u)^2}.\end{gathered} \tag{44}$$

Even in the simplest case $\xi^2 = 0$, when these equations can be cast into the form

$$a'' = -\frac{aa'}{u(a-u)} - \frac{2a^2a'^3}{(a-u)^2}, \quad \ddot{u} = \frac{a\dot{u}^2}{u(a-u)} + \frac{2a^2}{(a-u)^2}, \tag{45}$$

we have not succeeded in finding a general solution.

3. Asymptotics

However, we can investigate the asymptotics of exact physical solutions to the LD equation at large times. It is easily done in the coordinates where the vector field of the system (34) has no singularities. Making a change of variables

$$a = u\bar{a}, \quad b = u\bar{b}, \tag{46}$$

we come to

$$\begin{aligned} \lambda\dot{\bar{a}} &= \bar{a}[1 + 2\lambda\bar{b}(\xi^2 + u)] - \omega, \\ \lambda\dot{\bar{b}} &= \bar{b} - \lambda[\bar{a}^2 - \bar{b}^2(\xi^2 + 2u)], \\ \dot{u} &= -2\bar{b}u(\xi^2 + u). \end{aligned} \tag{47}$$

It is not difficult to find the stationary points of this system. We are interested only in physical solutions and physical stationary points. A physical stationary point as a particular case of a physical solution should be regular in λ. Again we have three cases.

The case $\xi^2 = 1$ is a planar motion in a constant homogeneous magnetic field [15,30]. The system (47) has two stationary points, one of them being physical,

$$\begin{gathered} \bar{a} = \frac{\omega}{g}, \quad \bar{b} = \frac{g-1}{2\lambda}, \quad u = 0, \\ g := 2^{-1/2}(1 + \sqrt{1 + 16\lambda^2\omega^2})^{1/2}. \end{gathered} \tag{48}$$

Linearizing the system (47) in the vicinity of this point, we obtain the asymptotics of the exact solution to the LD equation:

$$\begin{aligned} \delta\bar{a} &= -u(0)Ae^{\lambda^{-1}(1-g)\tau} + e^{\lambda^{-1}g\tau}\bigg[[c_1 + u(0)A]\cos\frac{2\omega\tau}{g} \\ &\quad + [c_2 + u(0)B]\sin\frac{2\omega\tau}{g}\bigg], \\ \delta\bar{b} &= -u(0)Be^{\lambda^{-1}(1-g)\tau} + e^{\lambda^{-1}g\tau}\bigg[[c_2 + u(0)B]\cos\frac{2\omega\tau}{g} \\ &\quad - [c_1 + u(0)A]\sin\frac{2\omega\tau}{g}\bigg], \\ \delta u &= u(0)e^{\lambda^{-1}(1-g)\tau}, \quad A := \frac{\omega(g-1)}{g(5g-4)}, \quad B := \frac{3(g-1)^2}{2\lambda(5g-4)}. \end{aligned} \tag{49}$$

The terms in the large square brackets describe runaway solutions. They are unphysical and have to be set to zero by a proper choice of the initial conditions. Then the physical solution to Eqs. (47) is solely specified by the initial conditions on the function $u(\tau)$. The first term in the first equation in Eqs. (49) describes a correction to the rotational speed of a charged particle due to the radiation reaction. Inasmuch as $u(\tau)$ is non-negative, this correction has the opposite sign with respect to the main contribution, i.e., the rotational speed increases with time and tends exponentially to its limiting value (48). The limiting value is, of course, less than the cyclotron frequency. In the case at hand, the function $u(\tau)$ is related to the kinetic energy of the particle and so the expression for $\delta u(\tau)$ in (49) describes its decrease.

The case $\xi^2 = -1$ corresponds to a planar motion in a constant homogeneous electric field. The system (47) possesses two stationary points. Only one of these points is physical,

$$\bar{a} = \omega, \quad \bar{b} = \frac{g-1}{2\lambda}, \quad u = 1, \quad g := \sqrt{1 + 4\lambda^2\omega^2}. \tag{50}$$

The LD equation linearized in the neighbourhood of this point has the solution

$$\begin{aligned} \delta\bar{a} &= -\omega u(0)(1-g^{-1})e^{\lambda^{-1}(1-g)\tau} + e^{\lambda^{-1}\tau}[c_1 + \omega u(0)(1-g^{-1})], \\ \delta\bar{b} &= -u(0)Be^{\lambda^{-1}(1-g)\tau} + \frac{2\lambda\omega}{g-1}e^{\lambda^{-1}\tau}\left[c_1 + \omega u(0)(1-g^{-1})\right] \\ &\quad + e^{\lambda^{-1}g\tau}\left[c_2 - \frac{2c_1\lambda\omega}{g-1} + \frac{u(0)(g-1)}{\lambda(1-2g)}\right], \\ \delta u &= u(0)e^{\lambda^{-1}(1-g)\tau}, \quad B := \frac{(g-1)^2(2g+1)}{2\lambda g(2g-1)}. \end{aligned} \tag{51}$$

The unphysical solutions are the terms in square brackets. Demanding their vanishing, we uniquely determine the integration constants c_1 and c_2 through $u(0)$. The correction to the "frequency" $\dot{\psi}(\tau)$ increases with time and tends exponentially to the limiting value (50). The expression for $\delta u(0)$ in (51) describes evolution of the square of the momentum component normal to the electric field. As expected, this component exponentially tends to zero and the solution passes into the hyperbolic motion (12).

The case $\xi^2 = 0$ is more involved. To shorten formulas, we redefine the variables entering (47):

$$\bar{a} \to \omega\bar{a}, \quad \bar{b} \to \lambda\omega^2\bar{b}, \quad u \to (\lambda\omega)^{-2}u, \quad \tau \to \lambda\tau, \tag{52}$$

and shall restore the original notation where it becomes necessary to make estimations. After this redefinition, a regularity in λ, which distinguishes physical solutions, means a regularity of the solution in τ^{-1}. Then the system (47) has a single stationary point

$$\bar{a} = 1, \quad \bar{b} = 1, \quad u = 0. \tag{53}$$

This point is degenerate and, therefore, the solutions to the linearized system improperly describe the behavior of solutions to the LD equation in the vicinity of this point. To obtain the correct asymptotics, we integrate the last equation in (47),

$$u = u(0)\left[1 + 2u(0)\tau + 2u(0)\int_0^\tau dt\,\delta\bar{b}(t)\right]^{-1}. \tag{54}$$

The integrand of the third term in the square brackets tends to zero. Consequently, the second term in the square brackets will dominate at large times,

$$\tau \gg \lambda, \quad \tau \gg \tau_{1c} := [2\lambda\omega^2u(0)]^{-1}, \tag{55}$$

and we can take

$$u \approx \tau^{-1}/2. \tag{56}$$

Then the equations for the leading asymptotics read as

$$\delta\dot{\bar a} = \delta\bar a + \tau^{-1}\delta\bar b + \tau^{-1}, \quad \delta\dot{\bar b} = -2\delta\bar a + \delta\bar b + \tau^{-1}, \tag{57}$$

where we keep only the leading terms. The system (57) possesses runaway solutions which are nonregular in λ (τ^{-1}) and, consequently, unphysical. They can be removed by an appropriate choice of the initial data. As for the physical solutions, their leading asymptotics takes the form

$$\begin{aligned}
\bar a &= 1 - \tau^{-1} - \tau^{-2}[3\ln\tau + 2u(1) - 6] + o(\tau^{-2}) \\
&= 1 - \frac{1}{\tau}\left(\frac{2u(1)\tau^3}{e^5}\right)^{1/\tau} + o(\tau^{-2}), \\
\bar b &= 1 - 3\tau^{-1} - \tau^{-2}[9\ln\tau + 6u(1) - 23] + o(\tau^{-2}) \\
&= 1 - \frac{3}{\tau}\left(\frac{2u(1)\tau^3}{e^{20/3}}\right)^{1/\tau} + o(\tau^{-2}), \\
u &= \frac{1}{2}\{\tau^{-1} + \tau^{-2}[3\ln\tau + 2u(1) - 1\} + o(\tau^{-2}) \\
&= \frac{1}{2\tau}[2u(1)\tau^3]^{1/\tau} + o(\tau^{-2}).
\end{aligned} \tag{58}$$

Here we also add a next-to-leading correction to the asymptotics, which can be derived from the initial nonlinear system (47). The last equalities in (58) show how the power of the proper time entering the asymptotics tends to its limiting value. The initial value $u(1)$ in these last formulas differs from $u(1)$ appearing in the second equalities in (58). These initial values are related in an evident manner.

Substituting the asymptotics (58) into Eqs. (42), (43) and bearing in mind the replacement (52), we find

$$\begin{aligned}
\upsilon_1 &= \left(\frac{\tau}{2\lambda}\right)^{1/2}\left(2\upsilon_1^2(\lambda)\frac{\tau}{\lambda}\right)^{\lambda/2\tau} + o((\lambda/\tau)^{1/2}), \\
\upsilon_0 &= \lambda\omega\left(\frac{\tau}{2\lambda}\right)^{3/2}\left(\frac{\lambda\omega^2\tau}{8\upsilon_0^2(\lambda)}\right)^{-\lambda/2\tau} + o((\tau/\lambda)^{1/2}) = -\upsilon_2.
\end{aligned} \tag{59}$$

As we see, the system goes to the universal regime. For example, the quantity

$$\frac{\upsilon_0}{\upsilon_1^3} \approx -\frac{\upsilon_2}{\upsilon_1^3} \approx \lambda\omega\left(\frac{\omega\upsilon_1^3(\lambda)\tau^2}{\lambda\upsilon_0(\lambda)}\right)^{-\lambda/\tau} \tag{60}$$

ceases to depend on the initial data and tends to $\lambda\omega$. This occurs on the proper time scales

$$\tau \gg \lambda\left|\ln\frac{\upsilon_0(\lambda)}{\lambda\omega\upsilon_1^3(\lambda)}\right|. \tag{61}$$

This effect is essentially due to the radiation reaction. After the lapse of a certain time, the charged particles moving on the plane in the electromagnetic field with the invariants $I_1 = I_2 = 0$ will have the same ratio of momenta, of the form (60), irrespective of their initial momenta. So, if we measure the momentum components of these charged particles, the measured data will lie on the cubic parabola determined by Eq. (60). It is clear that there is an infinite number of quantities depending on the momentum components which tend to some constant values at large proper times. It is also useful to consider the relation

$$2\lambda\omega p\upsilon_1 \approx \left(2p(\lambda)\upsilon_1(\lambda)\frac{\omega\tau^2}{\lambda}\right)^{\lambda/\tau}, \tag{62}$$

where we recall that $p = \upsilon_0 + \upsilon_2$. The combination of momentum components on the left-hand side tends to unity at large proper times.

For comparison, we give here the well-known solution to the Lorentz equation in this electromagnetic field:

$$\begin{aligned}
\upsilon_1 &= \upsilon_1(0) + \sqrt{u(0)}\omega\tau, \\
\upsilon_0 &= \upsilon_0(0) + \upsilon_1(0)\omega\tau + \sqrt{u(0)}\frac{\omega^2\tau^2}{2}, \\
\upsilon_2 &= \upsilon_2(0) - \upsilon_1(0)\omega\tau - \sqrt{u(0)}\frac{\omega^2\tau^2}{2}.
\end{aligned} \tag{63}$$

In this case, the quantity analogous to (60), which tends to a constant at large times, is the ratio

$$\frac{\upsilon_0}{\upsilon_1^2} \approx -\frac{\upsilon_2}{\upsilon_1^2} \approx \frac{1}{2\sqrt{u(0)}}. \tag{64}$$

Its limiting value depends on the initial conditions. The ratio (60) goes to zero as τ^{-1} with asymptotics depending on the initial data. As far as the relation (62) is concerned, it grows linearly with τ at large proper times in the solutions (63) to the Lorentz equation.

4. Landau-Lifshitz equation

Now we investigate the planar motion of a charged particle in a constant homogeneous electromagnetic field in the framework of the so-called Landau-Lifshitz equation [21]. This is an approximate equation describing the physical solutions to the LD equation. It is obtained from the LD equation by the reduction of order procedure, with λ being assumed to be a small parameter. Thus, let us seek for solutions to the LD equation (31) in a class of functions such that

$$\left|\frac{d^k m_\alpha}{d\tau^k}\right| = O(1), \quad \left|\frac{d^k\upsilon_\alpha}{d\tau^k}\right| = O(1), \quad k = \overline{0,\infty}, \tag{65}$$

with respect to the small parameter λ. Also, we restrict ourself to the first correction in λ to the Lorentz equation.

Differentiating the first equation in (31) with respect to τ, we find $\dot m_\alpha$. Then we substitute it into the initial equation and come to

$$m_\alpha = -\omega(\xi_\alpha - p\upsilon_\alpha - \lambda\dot p\upsilon_\alpha - \lambda p\dot\upsilon_\alpha). \tag{66}$$

By use of this relation, the second equation in (31) describing the evolution of the momentum υ_α is brought to (the Landau-Lifshitz equation)

$$\dot\upsilon_\alpha = \omega\varepsilon_{\alpha\beta\gamma}\xi^\beta\upsilon^\gamma + \lambda\omega^2 p(\xi_\alpha - p\upsilon_\alpha), \tag{67}$$

where we neglect the higher orders in λ. The solutions to this equation are known for the field configurations considered by us (see, e.g., [14,22,23]). Convolving the Landau-Lifshitz equation (67) with the vector ξ_α, we arrive at

$$\dot{u} = -2\lambda\omega^2 u(\xi^2 + u), \tag{68}$$

whence

$$u = \frac{\xi^2 u(0)}{[\xi^2 + u(0)]e^{2\lambda\omega^2\xi^2\tau} - u(0)}, \quad \xi^2 = \pm 1;$$
$$u = \frac{u(0)}{1 + 2\lambda\omega^2 u(0)\tau}, \quad \xi^2 = 0. \tag{69}$$

The Landau-Lifshitz equation has the form of Eq. (36). Hence, to describe the evolution of momentum components, we use the substitutions (37), (40), and (42). If $\xi^\alpha = (1,0,0)$, the substitution (37) gives

$$\dot{\varphi} = -\omega. \tag{70}$$

If $\xi^\alpha = (0,0,1)$, the substitution (40) results in

$$\dot{\psi} = \omega. \tag{71}$$

These formulas are in agreement with Eqs. (49) and (51) up to the first order in λ. In the isotropic case $\xi^\alpha = (1,0,1)$, we have from Eqs. (42) and (43)

$$\upsilon_1 = \frac{\upsilon_1(0) + \sqrt{u(0)}\omega\tau}{\sqrt{1 + 2\lambda\omega^2 u(0)\tau}}, \quad \upsilon_0 + \upsilon_2 = \frac{\sqrt{u(0)}}{\sqrt{1 + 2\lambda\omega^2 u(0)\tau}},$$
$$\upsilon_0 - \upsilon_2 = \frac{\upsilon_0(0) - \upsilon_2(0) + 2[\upsilon_1(0) + \lambda\omega\sqrt{u(0)}]\omega\tau + \sqrt{u(0)}\omega^2\tau^2}{\sqrt{1 + 2\lambda\omega^2 u(0)\tau}}. \tag{72}$$

We see that the system passes to the universal regime at sufficiently large proper times,

$$\upsilon_1 \approx \frac{\tau^{1/2}}{\sqrt{2\lambda}}, \quad \upsilon_0 + \upsilon_2 \approx \omega^{-1}\frac{\tau^{-1/2}}{\sqrt{2\lambda}}, \quad \upsilon_0 - \upsilon_2 \approx \omega\frac{\tau^{3/2}}{\sqrt{2\lambda}}. \tag{73}$$

The limiting values of the ratio (60) and the relation (62) are the same for these solutions as for the physical solutions to the exact LD equation. Thus, the Landau-Lifshitz equation correctly reproduces the asymptotics of the physical solution at large τ in spite of the fact that this asymptotics is not regular in λ.

The above procedure to integrate the Landau-Lifshitz equation is simply generalized to the case of a nonconstant external field parameter $\omega(\xi_\alpha x^\alpha(\tau))$. Despite the fact that an additional contribution to the first term on the right-hand side of Eq. (67) arises, the Landau-Lifshitz equation is still integrable by quadratures. For the isotropic case $\xi^2 = 0$ this solution is presented in [23].

With the explicit form of the solution (72) at hand, we can more accurately estimate the proper time needed for the ratio (60) and the relation (62) to become close to the respective asymptotic values. We have already found that the proper time τ must be much greater than the value τ_{1c} defined in Eq. (55). From the first equation in (72) we also see that it is necessary to demand

$$\tau \gg \tau_{2c} := |\upsilon_1(0)|/[p(0)\omega]. \tag{74}$$

In addition, in order to obtain the ratio (60) from the relations (73), we need

$$\tau \gg \tau_{3c} := \omega^{-1}. \tag{75}$$

Therefore, the ratio (60) holds at proper times much larger than τ_{1c}, τ_{2c}, and τ_{3c}, while the relation (62) is fulfilled for proper times much larger than τ_{1c} and τ_{2c} only. By increasing the initial energy of electrons and choosing an appropriate direction of the electron beam, we can diminish the characteristic proper times τ_{1c} and τ_{2c}, but the value of τ_{3c} is solely determined by the external field strength.

The trajectories of electrons $x^\mu(\tau)$ can be easily found from (72) in an analytic form. Some of their plots are presented on Fig. 1. Of course, in order that the particle moves along such a trajectory, it is sufficient to create a constant uniform electromagnetic field with vanishing invariants in a small vicinity of this trajectory and for the time needed for the electron to go along it. An analysis of the solutions $x^\mu(\tau)$ shows that, at large $u(0)$, the characteristic sizes of the trajectory (the distances from the origin to its turning points with respect to the axes) scale with the external field as $\omega^{-3/2}$. So, if the characteristic scale of the trajectory is approximately 1 km at the magnetic field strength $H = 10^4$ G (see Fig. 1), its size will be 1 m at $H = 10^6$ G.

The main obstacle to observation of the asymptotic behavior of electrons that we described consists in the fact that we cannot create, for the time being, very strong electromagnetic fields comparable with the Schwinger field (9) in large volumes. Hence, we cannot appreciably decrease the characteristic proper time τ_{3c}. As a result, the ratio (60) is hard to verify experimentally at the present moment, although it is possible. The asymptotic relation (62) is much easier to achieve provided that we take $\upsilon_1(0) \approx 0$ and increase the initial electron energy correspondingly. In the case $\upsilon_1(0) = 0$ and at high energies [large $p(0)$], we can simply estimate a distance from the origin to the point of the electron trajectory where the relation (62) approaches its limiting value

$$R \approx p(0)\tau_{1c}/2 = [4\lambda p(0)\omega^2]^{-1}. \tag{76}$$

The larger the initial Lorentz factor $\upsilon_0(0) \approx p(0)/2$, the smaller the distance R. This estimation is confirmed by the numerical results presented in Fig. 1.

5. *Stability of the asymptotics*

In the real situation, the electromagnetic field we can create is not ideally constant and uniform and with the exactly vanishing invariants $I_1 = I_2 = 0$. Therefore, it is advisable to analyze the stability of the asymptotics obtained in the case $\xi^2 = 0$ with respect to small perturbations of the external electromagnetic field. We want to deduce the constraint on the magnitude of these perturbations under which the charged particle has time to go to the universal regime. With this aim, we linearize the LD equation (7) in the neighborhood of its physical solutions with the asymptotics (59) or (73). Considering the linearized system of equations obtained at

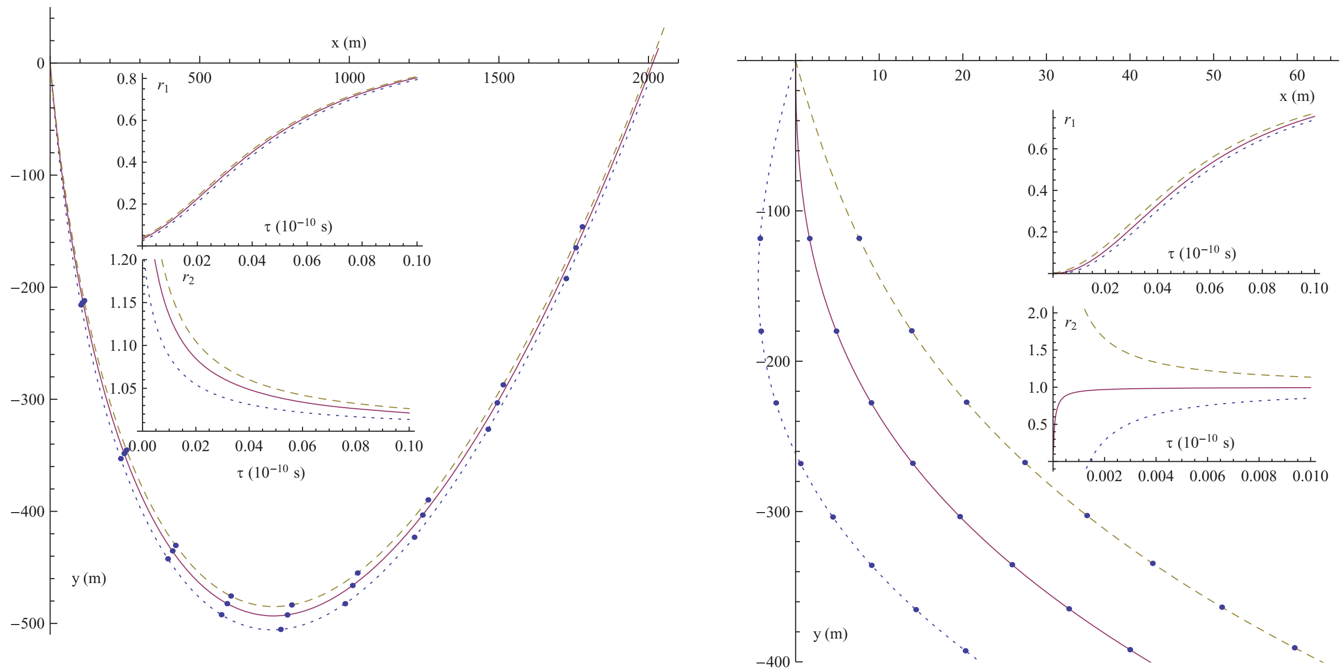

FIG. 1. (Color online) The trajectories of electrons launched from the origin in the crossed electromagnetic field $H_z = -E_x = 10^4$ G. The small plots depict the dependence of the ratios $r_1 := \lambda\omega\upsilon_1^3/\upsilon_0$ and $r_2 := 2\lambda\omega p\upsilon_1$ on the proper time τ. Left panel: The trajectories of electrons emitted from the origin at different angles with the Lorentz factor $\upsilon_0(0) = 10^6$ (energy $E \approx 511$ GeV). The blobs on the trajectories are located at the proper times $n \times 10^{-12}$ s. The characteristic proper times take the values $\tau_{1c} = 6.7 \times 10^{-13}$ s, $\tau_{2c} = 8.2 \times 10^{-13}$ s, and $\tau_{3c} = 5.7 \times 10^{-12}$ s. We see that the ratio r_2 tends to its asymptotic value faster than the ratio r_1 inasmuch as τ_{3c} is greater than τ_{1c} and τ_{2c}. Right panel: The trajectories of electrons launched from the origin with $\upsilon_1(0) \approx 0$ and the Lorentz factor $\upsilon_0(0) = 10^7$ (energy $E \approx 5.11$ TeV). The blobs on the trajectories are located at the proper times $n \times 10^{-13}$ s. The characteristic proper times take the values $\tau_{1c} = 6.4 \times 10^{-15}$ s, $\tau_{2c} = 0$ s, and $\tau_{3c} = 5.7 \times 10^{-12}$ s for the middle trajectory, and $\tau_{1c} = 6.4 \times 10^{-15}$ s, $\tau_{2c} = 1.4 \times 10^{-13}$ s, and $\tau_{3c} = 5.7 \times 10^{-12}$ s for the side trajectories. The distance (76) for the middle trajectory is approximately equal to 19 m.

sufficiently large proper times (55), we can write

$$\delta\ddot{x}_- = (\delta E_x + \delta H_z)\left(\frac{\tau}{2\lambda}\right)^{1/2} + \lambda\left(\delta\dddot{x}_- - \frac{\delta\ddot{x}_-}{8\lambda\tau}\right),$$

$$\begin{aligned}\delta\ddot{x}_3 &= \lambda\omega(\delta H_x - \delta E_z)\left(\frac{\tau}{2\lambda}\right)^{3/2} - \delta H_y\left(\frac{\tau}{2\lambda}\right)^{1/2} \\ &\quad + \lambda\left(\delta\dddot{x}_3 - \frac{\delta\dot{x}_3}{8\lambda\tau}\right), \\ \delta\ddot{x}_1 &= \omega\delta\dot{x}_- - \lambda\omega(\delta E_x + \delta H_z)\left(\frac{\tau}{2\lambda}\right)^{3/2} + \lambda\bigg[\delta\dddot{x}_1 \\ &\quad - \left(\frac{3\omega\tau}{4\lambda}\delta\ddot{x}_- + \frac{\delta\ddot{x}_1}{2\lambda}\right) - \frac{\delta\dot{x}_1}{8\lambda\tau}\bigg],\end{aligned} \tag{77}$$

where $\delta x_\mu(\tau)$ describes small perturbations of the trajectory of the charged particle due to the external field fluctuations $\delta F_{\mu\nu}(x)$. The equation for the fourth component of the perturbation δx_μ follows from the linearized mass-shell condition

$$\upsilon_\mu \delta\dot{x}^\mu = 0, \tag{78}$$

with $\upsilon_\mu(\tau)$ given by the asymptotics (73). The third-derivative terms entering the system (77) can be neglected as long as λ is small. Then the solution to the system of equations (77) is easy to write in terms of quadratures. Now we can compare the deviation of the four-momentum $\delta\dot{x}_\mu$ with the asymptotics (73). Roughly,

$$\delta\dot{x}/\upsilon \sim 8\delta F\omega\tau^2/13, \tag{79}$$

where δF denotes the magnitude of the perturbations of the external electromagnetic field components at the highest powers of τ in the system (77). We should demand that the quantity (79) is much less than unity for the asymptotics (59) to be observable. Combining this condition with the requirement (55), we come to the constraint on the fluctuations of the external electromagnetic field:

$$\delta F/F \ll 13\lambda^2\omega^2(\upsilon_0 + \upsilon_2)^2/2. \tag{80}$$

The parameter $\lambda\omega$ is the same as in Eq. (9). The requirements (74) and (75) should be fulfilled too. Thus, if these conditions are satisfied, there is a range of proper times when the physical solutions to the LD equation are close to the asymptotics (59) and the estimations for the ratio (60) and the relation (62) hold. At very large proper times, which are out of this range, the planar motion of a charged particle in an approximately constant and uniform electromagnetic field will end with the hyperbolic motion (12) (the case $\xi^2 = -1$) or with uniform rectilinear motion (the case $\xi^2 = 1$) since in general the magnitudes of the electric and magnetic field strengths are not equal to each other, $|\mathbf{E}| \neq |\mathbf{H}|$.

IV. DISCUSSION

In this paper, we have investigated planar motion of charged particles obeying the LD equation. We gave a detailed study of the asymptotics of planar physical solutions to the LD equation in a constant homogeneous electromagnetic field. One of the main results of our study is an interesting asymptotics of the physical solutions to the LD equation in this electromagnetic field with vanishing invariants $I_1 = I_2 = 0$. According to classical radiation reaction theory, charged particles moving on the plane in such an electromagnetic field must have the same ratios (60) and (62) of the momentum components at sufficiently large proper times and small fluctuations of the external electromagnetic field. The existence of the asymptotics (62) can be verified in a purely quantum electrodynamical context.

The Dirac equation in that electromagnetic field can be exactly solved [31]. Then, by the use of this complete set of solutions, we construct quantum electrodynamics on the given background [32] and define the S matrix. The operators of covariant momenta $\mathcal{P}_1 = m\upsilon_1$ and $\mathcal{P}_- = m(\upsilon_0 + \upsilon_2)$ commute. Therefore, we can construct a complete set of their eigenfunctions and evaluate the transition amplitudes to these states. For the one-photon radiation amplitude, the transition probability reads

$$\sum_{k,s} \langle \beta | \hat{U}^\dagger | \mathcal{P}_1, \mathcal{P}_-; k, s\rangle \langle \mathcal{P}_1, \mathcal{P}_-; k, s | \hat{U} | \beta \rangle, \tag{81}$$

where $\hat{U}$ is the evolution operator over an infinite time, β are the quantum numbers characterizing the initial state of the electron, and k and s are the momentum and polarization of the radiated photon. Provided the classical radiation reaction theory is viable, there must exist a range of quantum numbers β and strengths of the electromagnetic field such that the transition probability (81) is mostly concentrated on the hyperbola

$$\frac{4\alpha E}{3E_0} \frac{\mathcal{P}_1 \mathcal{P}_-}{m^2} = 1. \tag{82}$$

Of course, using formula (81), we disregard multiple-photon production and the production of electron-position pairs, but these contributions are proportional to the fine structure constant and negligible for reasonable strengths of the electromagnetic field. We postpone a detailed study of this effect in quantum electrodynamics for future research.

Also note that, as pointed out in [13], the notion of a physical solution seems to be quite general and can be exploited to give a proper interpretation of the higher-order derivative corrections to effective actions in quantum field theory (see, e.g., [33–35] and also their quantization in [36,37]). Usually, these terms arise from a heat kernel expansion over the regularization parameter Λ or large mass m of the one-loop correction [38] whereas the higher terms of this expansion are ignored. By analogy with the considerations presented in the Appendix, we should demand regularity of solutions to the effective equations of motion in the small expansion parameter—the inverse regularization parameter or inverse large mass. Then neglect of the nonlocal remainder of the effective action is justified. In many instances, this regularity can be related to the regularity with respect to the coupling constants, although it does not mean that we assume smallness of the couplings or these higher-order derivative terms. As far as the heat kernel expansion is concerned, one can distinguish corrections of three types: (i) divergent higher-derivative terms as, for example, in R^2 gravity; (ii) higher-derivative terms disappearing in the regularization removal limit; (iii) higher-derivative terms resulting from the large-mass expansion of a finite part of the effective action. In these cases, the coefficients of the higher-derivative terms have the forms

$$\text{(i) } \bar{\lambda}_a^{-1} + e_a(\bar{\lambda})/f_a(m/\Lambda), \quad \text{(ii) } e_b(\bar{\lambda}) f_b(m/\Lambda), \quad \text{(iii) } e_c(\bar{\lambda}) f_c(m^{-2}), \tag{83}$$

where e and f are some functions going to zero at the origin. The coefficients in the first case are the renormalized inverse couplings λ_a^{-1}. Now it is easy to see that regularity of the expression in the coupling constants λ_a implies its regularity in Λ^{-1}. As for the large-mass expansion, we additionally have to require regularity of solutions to the effective equations of motion in m^{-1}. Just to demonstrate what we mean, consider the effective action with a higher-derivative correction coming from the (self-)interaction

$$S[\phi] = \frac{1}{2} \int dx \phi(-\Box - m^2 + e\Box^2)\phi, \tag{84}$$

where e is proportional to the coupling constants or, possibly, to the inverse large mass. The physical sector of this model is equivalent to

$$S_{phys}[\phi] = \frac{1}{2} \int dx \phi[-\Box - (\sqrt{1 + 4em^2} - 1)/2e]\phi. \tag{85}$$

As long as as the coupling constants are scalars with respect to a symmetry group of the model, its physical sector possesses this symmetry as well. Elimination of the unphysical sector in free models is a simple task, but such an explicit elimination becomes complicated for fully interacting models and results in the appearance of nonlocal projectors to the physical states in the effective action.

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APPENDIX: REGULARITY CONDITION IN MAXWELL ELECTRODYNAMICS

In this appendix, we show that the solutions to the coupled system of Maxwell-Lorentz equations are regular in the coupling constants before we take the regularization removal limit.

Consider a model with the action functional (3). The effective equations of motion of a charged particle with a bare mass $\bar{m}$ look like

$$\bar{m}\ddot{x}_\mu(\tau)=eF_{\mu\nu}(x(\tau))\dot{x}^\nu(\tau)+4e^2\int d\tau'\theta(X^0(\tau,\tau'))\delta'(X^2(\tau,\tau') - \varepsilon)X_{[\mu}(\tau,\tau')\dot{x}_{\nu]}(\tau')\dot{x}^\nu(\tau), \tag{A1}$$

where $X_\mu(\tau,\tau') := x_\mu(\tau) - x_\mu(\tau')$, and we have used the regularization of the retarded Green's function of the form

$$G^-(x) = \frac{\theta(x^0)}{2\pi}\delta(x^2) \;\to\; G^-_\varepsilon(x) = \frac{\theta(x^0)}{2\pi}\delta(x^2-\varepsilon), \tag{A2}$$

where ε is a regularization parameter. We shall assume that $F_{\mu\nu}(x)$ is smooth and bounded on the spacetime, $F_{\mu\nu}(x(\tau))$ vanishes at $\tau < \tau_0$ for some τ_0, and the solution to Eq. (A1) has the following asymptotics in the past:

$$x_\mu(\tau) = (\tau-\tau_0)\upsilon_\mu + \bar{x}_\mu(\tau), \quad \upsilon^2 = 1, \tag{A3}$$

where $\bar{x}_\mu(\tau)$ is zero at $\tau < \tau_0$, and υ_μ is a constant four-vector. Unphysical solutions may appear on the scale of a classical electron radius. Therefore, we are looking for a solution of the form

$$x_\mu(\tau) = \bar{r}_e y_\mu(\tau/\bar{r}_e), \quad \bar{r}_e := e^2/\bar{m}. \tag{A4}$$

The function $y_\mu(s)$ also depends on other dimensionless combinations of parameters entering Eq. (A1), but we do not write them explicitly. Upon the substitution (A4), we arrive at

$$\ddot{y}_\mu(s) = \frac{e}{\bar{m}}\bar{r}_e F_{\mu\nu}(y(s))\dot{y}^\nu(s) + 4\int ds'\theta(Y^0(s,s'))\delta'(Y^2(s,s') - \bar{\varepsilon})Y_{[\mu}(s,s')\dot{y}_{\nu]}(s')\dot{y}^\nu(s), \tag{A5}$$

where $\bar{\varepsilon} := \varepsilon/\bar{r}_e^2$ and other evident redefinitions have been made. The integral in this equation can be taken:

$$I = \left[\frac{Y_{[\mu}(s,s')\ddot{y}_{\nu]}(s,s')}{[\dot{y}_\rho(s')Y^\rho(s,s')]^2} + \frac{Y_{[\mu}(s,s')\dot{y}_{\nu]}(s')}{[\dot{y}_\rho(s')Y^\rho(s,s')]^3} \times [1-\ddot{y}_\rho(s')Y^\rho(s,s')]\right]\dot{y}^\nu(s), \tag{A6}$$

where s' is determined by the conditions $Y^2(s,s') = \bar{\varepsilon}$, $Y^0(s,s') < 0$. The LD equation is obtained by expanding Eqs. (A5) and (A6) in asymptotic series in ε around zero and discarding the terms vanishing at $\varepsilon \to 0$. In view of the asymptotic behavior of the solution (A3), the asymptotics of the integral I at small $\bar{r}_e$ readily follows:

$$I \underset{\bar{r}_e\to 0}{\to} \varepsilon^{-3/2}\bar{r}_e^3\bar{y}_{[\mu(s)\upsilon_\nu]}\dot{y}^\nu(s). \tag{A7}$$

Hence, in this limit, the integro-differential equation (A1) reduces to a Lorentz-type differential equation with the effective strength tensor $F_{\mu\nu}$ plus the correction (A7). According to the general theorems of ordinary differential equation theory, the solutions to this equation are regular in $\bar{r}_e$. The effective strength tensor tends to zero when $\bar{r}_e \to 0$, and, in this limit, the particle moves along a straight line,

$$\ddot{y}_\mu(s) = 0. \tag{A8}$$

Restoring the notation (A4), we see that the limiting trajectory of the particle is indeed regular in $\bar{r}_e$. If we compel the Lorentz force to be constant at $\bar{r}_e \to 0$, increasing the strength of the electromagnetic field, the equation of motion (A5) turns into an ordinary Lorentz equation and its solutions cease to depend on $\bar{r}_e$. Another possibility is not to scale x_μ with $\bar{r}_e$ as in (A4), but to consider Eq. (A1) with solutions of the form $x^\mu(\tau/\bar{r}_e)$. With the same reasoning, it is easy to see that these solutions are regular in $\bar{r}_e$ as well. As long as

$$r_e = \frac{\bar{r}_e}{1+b\bar{r}_e/\varepsilon^{1/2}} = \frac{b^{-1}\varepsilon^{1/2}}{1+b^{-1}\varepsilon^{1/2}/\bar{r}_e}, \tag{A9}$$

where b is some constant, the regularity of a solution in $\bar{r}_e$ implies its regularity in the renormalized classical electron radius r_e and the regularization parameter $\varepsilon^{1/2}$.

If we used another regularization of the Green's function,

$$G^-_\varepsilon(x) = \frac{\theta(x^0)}{2\pi\varepsilon}\theta(x^2)g(x^2/\varepsilon), \quad \int_0^\infty dxg(x) = 1, \tag{A10}$$

then the asymptotics of the integral I would become

$$I \underset{\bar{r}_e\to 0}{\to} 4\varepsilon^{-3/2}\bar{r}_e^2\bar{y}_{[\mu}(s)\upsilon_{\nu]}\dot{y}^\nu(s)\int_0^\infty dtg'(t^2). \tag{A11}$$

The above arguments applied to this case reveal that there exists a regular limit $r_e \to 0$ of solutions to the effective equations of motion of a charged particle with the regularization (A10) as well. Regularity of the electromagnetic field generated by this charged particle is also obvious. The regularization of the Green's function is equivalent to the regularization of the current

$$j^\mu(x) \;\to\; j^\mu_\varepsilon(x) = \Box_x\int dyG^+_\varepsilon(x-y)j^\mu(y), \tag{A12}$$

where $j_\mu(x)$ is the current of a point charge. The regularization $G^+_\varepsilon(x)$ of the advanced Green's function is analogous to that of the retarded one (A10). The effective equations of motion of a charge with the regularized current have the form (A1), but with the "effective" Green's function (for details, see, e.g., [39])

$$G^{eff}_\varepsilon(x-y) := \int dzdz'\Box_x G^-_\varepsilon(x-z)G^-(z-z')\Box_{z'} G^+_\varepsilon(z'-y). \tag{A13}$$

It is not difficult to show that the effective regularized Green's function is zero in the past light cone. Poincaré invariance of this Green's function implies that

$$G^{eff}_\varepsilon(x) = \frac{\theta(x^0)}{2\pi\varepsilon}\theta(x^2)\tilde{g}(x^2/\varepsilon) + \varepsilon^{-1}\theta(-x^2)h(-x^2/\varepsilon), \tag{A14}$$

where $\tilde{g}(x)$ is a generalized function satisfying the condition (A10), while

$$\int_0^\infty dxh(x) = 0. \tag{A15}$$

The part of the effective Green's function (A14) with the support lying outside the light cone does not contribute to the integral in Eq. (A1). Thus, we revert to the case of the Green's function regularization considered above. Notice also that we can use the retarded Green's function of the form (A2) in the regularized current (A12) instead of the advanced Green's function. It can be proven that the effective Green's function takes the form (A14) in this case too.

When we pass from Eq. (A1) to the LD equation (5), we turn to a "truncated" description of a charge. Because of the truncation, spurious solutions arise which are nonregular in the coupling constants. They should be excluded, since only the regular solutions, which we call physical, to the LD equation are close to the solutions of (A1) at small regularization parameter ε.

Indeed, the LD equation is derived from (A1) by breaking off the series in $\varepsilon^{1/2}$. A solution regular in r_e is regular in $\varepsilon^{1/2}$. If we substitute such a solution of the LD equation into Eq. (A1) expanded in the series in $\varepsilon^{1/2}$, we shall ascertain that the obtained expression tends to zero with $\varepsilon \to 0$ for any τ. On the other hand, if we substitute a nonregular solution of the LD equation into Eq. (A1) then the limit $\varepsilon \to 0$ of the obtained expression may not exist. Making a general ansatz nonregular in r_e for the LD equation (5), it is not difficult to see that the nonregular solutions to the LD equation have to possess an essentially singular point at $r_e = 0$ provided some miraculous cancellations do not occur. Since it is the LD force which is responsible for the singularity, it should dominate over the Lorentz force at small r_e and certain proper times τ. Therefore, the nonregular solution to the LD equation tends to the nonregular solution (14) to the free LD equation at $r_e \to 0$. This can be directly verified from (5) by stretching the proper time $\tau \to r_e\tau$. The solution (14) does have the essentially singular point at $r_e = 0$. But if we substitute this solution into Eq. (A1), it blows up at $\varepsilon \to 0$ (the regularization parameter also enters the renormalized mass). So the nonregular solutions cannot be regarded as approximate solutions to (A1) at small regularization parameter ε.

[1] P. A. M. Dirac, Proc. R. Soc. London, Ser. A **167**, 148 (1938).
[2] C. Teitelboim, Phys. Rev. D **1**, 1572 (1970); **3**, 297 (1971); **4**, 345 (1971); P. E. G. Rowe, *ibid*. **18**, 3639 (1978); K. Lechner and P. A. Marchetti, Ann. Phys. (N.Y.) **322**, 1162 (2007); K. Lechner, J. Phys. A **39**, 11647 (2006).
[3] V. Krivitskii and V. Tsytovich, Sov. Phys. Usp. **34**, 250 (1991); Ph. R. Johnson and B. L. Hu, Phys. Rev. D **65**, 065015 (2002); e-print arXiv:quant-ph/0012137; e-print arXiv:quant-ph/0012135.
[4] N. P. Klepikov, Sov. Phys. Usp. **28**, 506 (1985).
[5] C. S. Shen, Phys. Rev. D **17**, 434 (1978).
[6] V. G. Bagrov, G. S. Bisnovatyi-Kogan, V. A. Bordovitsyn, A. V. Borisov, O. F. Dorofeev, V. Ya. Epp, V. S. Gushchina, and V. Ch. Zhukovsky, *Synchrotron Radiation Theory and Its Development* (World Scientific, Singapore, 1999).
[7] J. Frenkel, Z. Phys. **37**, 243 (1926); H. J. Bhabha and H. C. Corben, Proc. R. Soc. London, Ser. A **178**, 273 (1941); A. O. Barut and N. Unal, Phys. Rev. A **40**, 5404 (1989); P. O. Kazinski, Zh. Eksp. Teor. Fiz. **132**, 370 (2007) [J. Exp. Theor. Phys. **105**, 327 (2007)].
[8] S. K. Wong, Nuovo Cimento **65**, 689 (1970); B. P. Kosyakov, Sov. Phys. Usp. **35**, 135 (1992).
[9] B. S. DeWitt and R. W. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960); J. M. Hobbs, *ibid*. **47**, 141 (1968); D. V. Gal'tsov and P. Spirin, Gravit. Cosmol. **13**, 241 (2007).
[10] B. P. Kosyakov, Teor. Mat. Fiz. **119**, 119 (1999) [Theor. Math. Phys. **199**, 493 (1999)]; P. O. Kazinski, S. L. Lyakhovich, and A. A. Sharapov, Phys. Rev. D **66**, 025017 (2002).
[11] F. Rohrlich, Phys. Rev. **150**, 1104 (1966); J. R. VanMeter, A. K. Kerman, P. Chen, and F. V. Hartemann, Phys. Rev. E **62**, 8640 (2000).
[12] P. O. Kazinski and A. A. Sharapov, Class. Quantum Grav. **20**, 2715 (2003).
[13] H. J. Bhabha, Phys. Rev. **70**, 759 (1946).
[14] N. D. Sen Gupta, Int. J. Theor. Phys. **4**, 179 (1971); **4**, 389 (1971); **8**, 301 (1973); Phys. Rev. D **5**, 1546 (1972).
[15] G. N. Plass, Rev. Mod. Phys. **33**, 37 (1961).
[16] A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Dover, New York, 1964).
[17] F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1965).
[18] H. Spohn, *Dynamics of Charged Particles and Their Radiation Field* (Cambridge University Press, Cambridge, 2004).
[19] A. A. Sokolov and I. M. Ternov, *Radiation from Relativistic Electrons* (American Institute of Physics, New York, 1986).
[20] F. Denef, J. Raeymaekers, U. M. Studer, and W. Troost, Phys. Rev. E **56**, 3624 (1997).
[21] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1962).
[22] J. C. Herrera, Phys. Rev. D **7**, 1567 (1973).
[23] A. Di Piazza, Lett. Math. Phys. **83**, 305 (2008).
[24] H. A. Lorentz, *Theory of Electrons* (B. G. Teubner, Leipzig, 1909).
[25] V. V. Golubev, *Lectures on the Analytic Theory of Differential Equations* (Gostekhizdat, Moscow, 1950) (in Russian).
[26] G. H. Hardy, *Divergent Series* (Clarendon Press, Oxford, 1949).
[27] P. O. Kazinski, Phys. Rev. E **77**, 041119 (2008).
[28] A. T. Fomenko, B. A. Dubrovin, and S. P. Novikov, *Modern Geometry: Methods and Applications. Vol. 1: The Geometry of Surfaces, Transformation Groups, and Fields* (Springer, New York, 1984)
[29] M. S. Plyushchay, Mod. Phys. Lett. A **10**, 1463 (1995).
[30] D. J. Endres, Nonlinearity **6**, 953 (1993).
[31] V. G. Bagrov, D. M. Gitman, *Exact Solutions of Relativistic Wave Equations* (Kluwer Academic, Dordrecht, 1990).
[32] E. S. Fradkin, D. M. Gitman, and Sh. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer, Berlin, 1991).
[33] X. Jaen, J. Llosa, and A. Molina, Phys. Rev. D **34**, 2302 (1986).
[34] R. P. Woodard, in *The Invisible Universe: Dark Matter and Dark Energy*, edited by L. Papantonopoulos, Lecture Notes in Physics, Vol. 720 (Springer, Berlin, 2007), p. 403.
[35] A. Morozov, Theor. Math. Phys. **157**, 1542 (2008).
[36] D. M. Gitman, S. L. Lyakhovich, and I. V. Tyutin, Russ. Phys. J. **26**, 730 (1983); S. L. Lyakhovich, Ph.D. thesis, Tomsk State University, 1985 (in Russian); I. L. Buchbinder and S. L. Lyahovich, Class. Quantum Grav. **4**, 1487 (1987).
[37] M. S. Plyushchay, Int. J. Mod. Phys. A **4**, 3851 (1989); Electron. J. Theor. Phys. **3**, 17 (2006).
[38] D. V. Vassilevich, Phys. Rep. **388**, 279 (2003).
[39] P. O. Kazinski and A. A. Sharapov, Teor. Mat. Fiz. **143**, 375 (2005) [Theor. Math. Phys. **143**, 798 (2005)].