An introduction to the Lorentz-Dirac equation

Eric Poisson
Department of Physics, University of Guelph, Guelph, Ontario, Canada N1G 2W1

These notes provide two derivations of the Lorentz-Dirac equation. The first is patterned after Landau and Lifshitz and is based on the observation that the half-retarded minus half-advanced potential is entirely responsible for the radiation-reaction force. The second is patterned after Dirac, and is based upon considerations of energy-momentum conservation; it relies exclusively on the retarded potential. The notes conclude with a discussion of the difficulties associated with the interpretation of the Lorentz-Dirac equation as an equation of motion for a point charge. The presentation is essentially self-contained, but the reader is assumed to possess some elements of differential geometry (necessary for the second derivation only).

arXiv:gr-qc/9912045v1 10 Dec 1999

I. INTRODUCTION

The Lorentz-Dirac equation is an equation of motion for a charged particle under the influence of an external force as well as its own electromagnetic field. The particle's world line is described by the relations $z^\alpha(\tau)$, which give the particle's coordinates as functions of proper time. We let $u^\alpha(\tau) = dz^\alpha/d\tau$ be the four-velocity, and $a^\alpha(\tau) = du^\alpha/d\tau$ is the four-acceleration. The Lorentz-Dirac equation is given by

$$ma^\alpha = F^\alpha_{\rm ext} + \frac{2}{3}\, q^2 \big(\delta^\alpha_{\ \beta} + u^\alpha u_\beta\big)\dot{a}^\beta, \tag{1.1}$$

where m is the particle's mass, q its charge, $F^\alpha_{\rm ext}$ the external force, and $\dot{a}^\alpha = da^\alpha/d\tau$. Throughout the paper we work in relativistic units, so that the speed of light is equal to unity. In a Lorentz frame in which the particle is momentarily at rest at the time τ, Eq. (1.1) reduces to

$$m\boldsymbol{a} = \boldsymbol{F}_{\rm ext} + \frac{2}{3}\, q^2 \dot{\boldsymbol{a}}, \tag{1.2}$$

where $\dot{\boldsymbol{a}} = d\boldsymbol{a}/dt$; bold-faced symbols denote the spatial components of the corresponding four-vectors. The special Lorentz frame in which Eq. (1.2) is valid will be referred to as the "momentarily comoving Lorentz frame", or MCLF. Equation (1.2) also gives the nonrelativistic limit of the Lorentz-Dirac equation.

My main objective with these notes is to provide a self-contained derivation of the Lorentz-Dirac equation. In fact, I will present two such derivations, each of which relying on a different set of heuristic assumptions. The first derivation is patterned after the presentation of Landau and Lifshitz [1], which I extend further. The second derivation is patterned after Dirac's classic paper [2], although it differs from it in its technical aspects — my own presentation requires much less calculational labour. The level of rigour achieved in these notes matches that of those two sources. It does not, however, match the level achieved in what I consider to be the best reference on this topic, the 1980 review article by Teitelboim, Villarroel, and van Weert [3]. Those three references were my main sources of inspiration. I have also relied on the books by Jackson [4] and Rohrlich [5].

I begin with a review of the fundamental equations of electromagnetism in Sec. II. In Sec. III I present the Landau-Lifshitz derivation of the nonrelativistic Lorentz-Dirac equation, Eq. (1.2). This reveals that the radiation-reaction part of the vector potential is given by $\frac{1}{2}(A^\alpha_{\rm ret} - A^\alpha_{\rm adv})$, where $A^\alpha_{\rm ret}$ is the retarded solution to the wave equation satisfied by the vector potential, while $A^\alpha_{\rm adv}$ is the advanced solution. The remaining part of the (retarded) potential, $\frac{1}{2}(A^\alpha_{\rm ret} + A^\alpha_{\rm adv})$, does not affect the motion of the charged particle.

Solving the wave equation for $A^\alpha_{\rm ret}(x)$ is handled via a Green's function whose entire support is on the past light cone of the field point x. This introduces a mapping between x and a specific point z on the particle's world line, the point at which it intersects x's past light cone. The mathematical aspects of this mapping are developed in Sec. IV, in preparation for Sec. V, in which the potential and field of a point charge are calculated. The Landau-Lifshitz calculation of the radiation-reaction force is resumed in Sec. VI, this time in a fully relativistic context. The calculation is once again based on the half-retarded minus half-advanced potential, and it produces Eq. (1.1).

The potential and field at x are naturally expressed in terms of u, the particle's proper time when it encounters x's past light cone, and r, the distance between x and the particle, as measured at that time in the MCLF. This motivates the introduction of a (noninertial) coordinate system for flat spacetime, based on u, r, and two polar angles. This new system is centered on the accelerated world line, and Sec. VII is devoted to its explicit construction. This prepares the way for Dirac's derivation of Eq. (1.1), which is presented in Sec. VIII. Dirac's derivation is based upon considerations of energy-momentum conservation, and it involves only the retarded potential.

The fact that the Lorentz-Dirac equation involves the derivative of the acceleration vector introduces considerable difficulties in its interpretation as an equation of motion. These difficulties are reviewed in Sec. IX; the point of view developed there comes largely from the excellent discussion contained in Flanagan and Wald [6]. These difficulties stem from the basic observation that point particles cannot be given a fully consistent treatment in a classical theory of electromagnetism. As a way of resolving these difficulties, Landau and Lifshitz take the Lorentz-Dirac equation through a reduction-of-order procedure, in which q^2/m is formally treated as a small quantity. This procedure is motivated and described in detail in Sec. IX. It produces a modified Lorentz-Dirac equation,

$$ma^\alpha = F^\alpha_{\rm ext} + \frac{2}{3}\frac{q^2}{m}\left(\delta^\alpha_{\ \beta} + u^\alpha u_\beta\right) F^\beta_{{\rm ext}\,,\gamma} u^\gamma, \qquad (1.3)$$

which formally is equivalent to Eq. (1.1), but is free of difficulties. In the MCLF, or in the nonrelativistic limit, Eq. (1.3) reduces to

$$m\boldsymbol{a} = \boldsymbol{F}_{\rm ext} + \frac{2}{3}\frac{q^2}{m}\dot{\boldsymbol{F}}_{\rm ext}, \qquad (1.4)$$

where $\dot{\boldsymbol{F}}_{\rm ext}$ is the complete (convective) time derivative of the external force.

II. THE FUNDAMENTAL EQUATIONS

We consider an electromagnetic field $F_{\alpha\beta}$ produced by a point charge q moving in flat spacetime on a world line $z^\alpha(\tau)$, where τ is proper time. The corresponding current density j^α is given by

$$j^\alpha(x) = q\int d\tau\, u^\alpha\, \delta(x-z), \qquad (2.1)$$

where $u^\alpha(\tau) = dz^\alpha/d\tau$ is the particle's four-velocity; the integration is over the complete world line, and the δ-function is four dimensional. Initially we take the world line to be arbitrary; our main goal is to find the particle's equations of motion.

The Maxwell field equations are

$$F^{\alpha\beta}_{\quad\ ,\beta} = 4\pi j^\alpha. \qquad (2.2)$$

If we express the electromagnetic field in terms of a vector potential,

$$F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}, \qquad (2.3)$$

and if we adopt the Lorentz gauge, $A^\alpha_{\ ,\alpha} = 0$, then the field equations take the simple form

$$\Box A^\alpha = -4\pi j^\alpha, \qquad (2.4)$$

in which $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ is the wave operator. The Minkowski metric diag$(-1,1,1,1)$ is denoted $\eta_{\alpha\beta}$, and $\eta^{\alpha\beta}$ is its inverse.

This equation can be solved with the help of a Green's function $G(x,x')$, which satisfies

$$\Box G(x,x') = -4\pi\delta(x-x'); \qquad (2.5)$$

the solution is $A^\alpha(x) = \int G(x,x') j^\alpha(x')\, d^4x' + A^\alpha_{\rm hom}(x)$, in which the second term represents a solution to the homogeneous equation. The *retarded* Green's function can be expressed as

$$G_{\rm ret}(x,x') = \theta(t-t')\,\delta(\sigma) \qquad (2.6)$$

$$= \frac{\delta(t-t'-|\boldsymbol{x}-\boldsymbol{x'}|)}{|\boldsymbol{x}-\boldsymbol{x'}|}. \qquad (2.7)$$

In the first expression, the quantity σ is defined by

$$\sigma(x,x') = \frac{1}{2}\,\eta_{\alpha\beta}(x-x')^\alpha(x-x')^\beta; \qquad (2.8)$$

it is half the spacetime interval between the points x and x'. The retarded Green's function has support only on the past light cone of the field point x. The second expression is obtained from the first by factorizing σ; $\boldsymbol{x}$ is the spatial projection of the four-vector x^α and $|\boldsymbol{x}|$ is its magnitude. The *advanced* Green's function is given by

$$G_{\rm adv}(x,x') = \theta(t'-t)\delta(\sigma) \qquad (2.9)$$

$$= \frac{\delta(t-t'+|\boldsymbol{x}-\boldsymbol{x'}|)}{|\boldsymbol{x}-\boldsymbol{x'}|}. \qquad (2.10)$$

It has support on the future light cone of the field point x.

Problem 1. Show that Eqs. (2.6)–(2.10) give the correct expressions for the retarded and advanced Green's functions.

The field equations (2.2) imply the charge-conservation equation $j^\alpha_{\ ,\alpha} = 0$. It is easy to check that this equation is satisfied by the current density of Eq. (2.1); the calculation requires the identity

$$u^\alpha\partial_\alpha\delta(x-z) = -\frac{d}{d\tau}\,\delta(x-z), \qquad (2.11)$$

which is most easily established in the particle's momentarily comoving Lorentz frame (MCLF) — the frame in which the particle is momentarily at rest at the time τ. Another relevant conservation equation is

$$(T^{\alpha\beta}_{\rm em} + T^{\alpha\beta}_{\rm part})_{,\beta} = 0, \qquad (2.12)$$

where

$$T^{\alpha\beta}_{\rm em} = \frac{1}{4\pi}\left(F^{\alpha\mu}F^\beta_{\ \mu} - \frac{1}{4}\,g^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}\right) \qquad (2.13)$$

is the electromagnetic field's stress-energy tensor, while

$$T^{\alpha\beta}_{\rm part} = m\int d\tau\, u^\alpha u^\beta\,\delta(x-z) \qquad (2.14)$$

is the stress-energy tensor of a point particle with mass m.

It is easy to show that

$$\partial_\beta T^{\alpha\beta}_{\rm em} = -F^\alpha_{\ \beta}\, j^\beta = -q \int d\tau\, F^\alpha_{\ \beta} u^\beta\, \delta(x-z), \qquad (2.15)$$

while

$$\partial_\beta T^{\alpha\beta}_{\rm part} = m \int d\tau\, a^\alpha\, \delta(x-z), \qquad (2.16)$$

where $a^\alpha(\tau) = du^\alpha/d\tau$ is the particle's acceleration. Substituting Eq. (2.15) and (2.16) into Eq. (2.12) yields the Lorentz-force equation,

$$m a^\alpha = q F^\alpha_{\ \beta} u^\beta. \qquad (2.17)$$

In this equation, $F_{\alpha\beta}$ must be evaluated at the point $x = z(\tau)$. Since $F_{\alpha\beta}$ is obviously singular on the world line, Eq. (2.17) is not very meaningful as it stands. Our aim is therefore to make sense of this equation.

Problem 2. Show that Eqs. (2.2) and (2.17) follow from the following action principle:

$$S = \int \left(-\frac{1}{16\pi}\, F_{\alpha\beta} F^{\alpha\beta} + A_\alpha j^\alpha \right) d^4x - m \int d\tau.$$

Here, $j^\alpha = q \int d\lambda\, \dot{z}^\alpha\, \delta(x-z)$ and $d\tau = (-g_{\alpha\beta}\dot{z}^\alpha \dot{z}^\beta)^{1/2}\, d\lambda$, where $\dot{z}^\alpha = dz^\alpha/d\lambda$; λ is an arbitrary parameter on the world line.

In a specific Lorentz frame, A^α can be decomposed into a scalar potential Φ and a vector potential $\boldsymbol{A}$. Similarly, $F_{\alpha\beta}$ can be decomposed into an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$. In this Lorentz frame, Eq. (2.3) becomes

$$\boldsymbol{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \boldsymbol{A}}{\partial t}, \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \overset{\vec{A}}{\cancel{\boldsymbol{B}}}, \qquad (2.18)$$

and Eq. (2.17) reduces to

$$m\boldsymbol{a} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \qquad (2.19)$$

where $\boldsymbol{v} = d\boldsymbol{z}/dt$ and $\boldsymbol{a} = d\boldsymbol{v}/dt$. Also, j^α can be decomposed into a charge density ρ and a current density $\boldsymbol{j}$; these are obtained by changing the variable of integration in Eq. (2.1) to z^0:

$$\rho(t,\boldsymbol{x}) = q\, \delta(\boldsymbol{x}-\boldsymbol{z}), \qquad \boldsymbol{j}(t,\boldsymbol{x}) = q\boldsymbol{v}\, \delta(\boldsymbol{x}-\boldsymbol{z}). \qquad (2.20)$$

In these expressions, $\boldsymbol{z}$ and $\boldsymbol{v}$ are considered to be functions of t.

III. RADIATION REACTION FOR SLOWLY MOVING CHARGES

The assumption that the charge is moving slowly gives rise to a clear identification of the radiation-reaction part of the electromagnetic field $F_{\alpha\beta}$. In this section we will show that the part of the vector potential which is responsible for the radiation reaction is

$$A^\alpha_{\rm rr}(x) = \frac{1}{2}\left[A^\alpha_{\rm ret}(x) - A^\alpha_{\rm adv}(x)\right]. \qquad (3.1)$$

The remaining part, $\frac{1}{2}(A^\alpha_{\rm ret} + A^\alpha_{\rm adv})$, must be associated with the particle's Coulomb field and does not influence the particle's motion. The two parts add up to $A^\alpha_{\rm ret}$, the correct (retarded) solution to the Maxwell field equations. The radiation-reaction potential (3.1) gives rise to a radiation-reaction field $F^{\alpha\beta}_{\rm rr}$, and it is this quantity that must be substituted to the right-hand side of Eq. (2.17) to obtain the correct equations of motion. Our considerations in this section will be limited to the nonrelativistic limit; the general case will be considered in the following sections.

Using Eqs. (2.7) and (2.10), we find that the retarded and advanced solutions to Eq. (2.4) are

$$A^\alpha_\epsilon(t,\boldsymbol{x}) = \int \frac{j^\alpha(t-\epsilon|\boldsymbol{x}-\boldsymbol{x'}|, \boldsymbol{x'})}{|\boldsymbol{x}-\boldsymbol{x'}|}\, d^3x', \qquad (3.2)$$

up to the possible addition of a solution to the homogeneous equation; such a term would give rise to an external field $F^{\alpha\beta}_{\rm ext}$ which can always be introduced at a later stage. The parameter ϵ is equal to $+1$ for the retarded solution, and to -1 for the advanced solution. For the time being we leave j^α arbitrary; it could describe an extended charge distribution. We assume that this charge distribution moves slowly: Let r_c be a characteristic length scale and t_c a characteristic time scale associated with the source's motion; we assume $v \equiv r_c/t_c \ll 1$. We will use this condition to simplify the appearance of Eq. (3.2).

Suppose first that we want to evaluate A^α_ϵ far from the source. Under the condition $r \equiv |\boldsymbol{x}| \gg |\boldsymbol{x'}|$, we have $|\boldsymbol{x}-\boldsymbol{x'}| = r - \boldsymbol{n}\cdot\boldsymbol{x'} + O(r^{-1})$, where $\boldsymbol{n} = \boldsymbol{x}/r$ is a unit vector pointing in the direction of $\boldsymbol{x}$. If we now expand j^α in a Taylor series about $t - \epsilon r$, treating $\boldsymbol{n}\cdot\boldsymbol{x'}$ as a small quantity, we obtain

$$j^\alpha(t-\epsilon|\boldsymbol{x}-\boldsymbol{x'}|, \boldsymbol{x'}) = \sum_{l=0}^{\infty} \frac{\epsilon^l}{l!}\, (\boldsymbol{n}\cdot\boldsymbol{x'})^l\, \frac{\partial^l}{\partial w^l}\, j^\alpha(w, \boldsymbol{x'}),$$

where $w = t - \epsilon r$ represents retarded time if $\epsilon = 1$ and advanced time if $\epsilon = -1$. The slow-motion approximation ensures than in this sum, the contribution at order l is a factor $(r_c/t_c)^l = v^l$ smaller than the leading term ($l = 0$). Substituting this expression for j^α into Eq. (3.2) yields a multipole expansion for the vector potential:

$$A^\alpha_\epsilon(t,\boldsymbol{x}) = \frac{1}{r} \sum_{l=0}^{\infty} \frac{\epsilon^l}{l!}\, n_L\, \frac{d^l}{dw^l} \int j^\alpha(w, \boldsymbol{x'})\, x'^L\, d^3x', \qquad (3.3)$$

where we use a multi-index L to denote a product of l identical factors. For example, $n_L = n_{a_1} n_{a_2} \cdots n_{a_l}$, and summation over a repeated multi-index is understood. If $\epsilon = 1$, then rA^α is a function of $t - r$ and the vector potential represents an *outgoing* wave; to such a wave corresponds an outward flux of energy, and outgoing waves therefore *remove* energy from the source. On the other hand, if $\epsilon = -1$ then rA^α is a function of $t+r$ and the vector potential represents an *ingoing* wave; to such a wave corresponds an inward flux of energy, and ingoing waves therefore *provide* energy to the source. This distinction between the retarded ($\epsilon = 1$) and advanced ($\epsilon = -1$) solutions is very important for radiation reaction.

Problem 3. First, show that to leading order in a slow-motion approximation, Eq. (3.3) reduces to

$$r\Phi_\epsilon(t,\boldsymbol{x}) = q + \epsilon\, \boldsymbol{n}\cdot\dot{\boldsymbol{p}}, \qquad r\boldsymbol{A}_\epsilon(t,\boldsymbol{x}) = \dot{\boldsymbol{p}},$$

where $q = \int \rho\, d^3x$ is the total charge and $\boldsymbol{p} = \int \rho\, \boldsymbol{x}\, d^3x$ the dipole moment of the charge distribution. The total charge is a constant, and $\boldsymbol{p}$ is a function of $w = t - \epsilon r$; an overdot denotes differentiation with respect to w. Second, show that the magnetic field is given by

$$r\boldsymbol{B}_\epsilon = -\epsilon\, \boldsymbol{n}\times\ddot{\boldsymbol{p}},$$

and that the electric field satisfies $\boldsymbol{E} = \epsilon\, \boldsymbol{B}\times\boldsymbol{n}$. Third, calculate the Poynting vector and show that the rate at which energy is flowing out of a sphere of radius r is given by

$$\frac{dE}{dw} = \epsilon\, \frac{2}{3}\, \ddot{\boldsymbol{p}}^2.$$

Thus, we have an outward flux if $\epsilon = 1$ and an inward flux if $\epsilon = -1$. For a point charge, $\boldsymbol{p} = q\boldsymbol{z}$ and $dE/dw = \epsilon\, \frac{2}{3} q^2 \boldsymbol{a}^2$. This is the well-known Larmor formula.

Suppose now that we want to evaluate A^α_ϵ inside the source. In such a situation we may expand j^α in a Taylor series about t, treating $|\boldsymbol{x}-\boldsymbol{x'}|$ as a small quantity. This gives

$$j^\alpha(t-\epsilon|\boldsymbol{x}-\boldsymbol{x'}|,\boldsymbol{x'}) = \sum_{l=0}^{\infty} \frac{(-\epsilon)^l}{l!}\, |\boldsymbol{x}-\boldsymbol{x'}|^l\, \frac{\partial^l}{\partial t^l}\, j^\alpha(t,\boldsymbol{x'}),$$

and again we see that the contribution at order l is a factor v^l smaller than the leading term. In this expression we notice that $(-\epsilon)^l = +1$ if l is even, while $(-\epsilon)^l = -\epsilon$ if l is odd. Substituting this into Eq. (3.2) gives

$$\begin{aligned} A^\alpha_\epsilon(t,\boldsymbol{x}) = &\sum_{l\ \mathrm{even}} \frac{1}{l!}\, \frac{\partial^l}{\partial t^l} \int j^\alpha(t,\boldsymbol{x'})\, |\boldsymbol{x}-\boldsymbol{x'}|^{l-1}\, d^3x' \\ &- \epsilon \sum_{l\ \mathrm{odd}} \frac{1}{l!}\, \frac{\partial^l}{\partial t^l} \int j^\alpha(t,\boldsymbol{x'})\, |\boldsymbol{x}-\boldsymbol{x'}|^{l-1}\, d^3x'. \end{aligned}$$

In this expression, the first sum gives $\frac{1}{2}(A^\alpha_{\mathrm{ret}} + A^\alpha_{\mathrm{adv}})$, and this contribution to A^α_ϵ is the same irrespective of the nature of the radiation at infinity, that is, whether the waves are outgoing or ingoing. This contribution to the vector potential cannot be responsible for the radiation reaction. On the other hand, the second sum gives $\frac{1}{2}(A^\alpha_{\mathrm{ret}} - A^\alpha_{\mathrm{adv}})$, and this contribution to A^α_ϵ changes sign under a change of boundary conditions at infinity. It is clearly this contribution to the vector potential that is responsible for the radiation reaction.

We therefore conclude that the radiation-reaction potential is indeed given by Eq. (3.1). In a slow-motion approximation, this reduces to

$$A^\alpha_{\mathrm{rr}}(t,\boldsymbol{x}) = -\sum_{l\ \mathrm{odd}} \frac{1}{l!}\, \frac{\partial^l}{\partial t^l} \int j^\alpha(t,\boldsymbol{x'})\, |\boldsymbol{x}-\boldsymbol{x'}|^{l-1}\, d^3x'. \tag{3.4}$$

The leading contribution to the scalar potential comes from $l = 3$ — the $l = 1$ term vanishes identically by virtue of charge conservation. Thus,

$$\Phi_{\mathrm{rr}}(t,\boldsymbol{x}) = -\frac{1}{3!}\, \frac{\partial^3}{\partial t^3} \int \rho(t,\boldsymbol{x'})\, |\boldsymbol{x}-\boldsymbol{x'}|^2\, d^3x'.$$

The leading contribution to the vector potential comes from $l = 1$:

$$\boldsymbol{A}_{\mathrm{rr}}(t,\boldsymbol{x}) = -\frac{\partial}{\partial t} \int \boldsymbol{j}(t,\boldsymbol{x'})\, d^3x'.$$

Both expressions neglect terms which are smaller by factors of order v^2.

We now specialize to the case of a point charge, and use Eqs. (2.20). We obtain

$$\Phi_{\mathrm{rr}}(t,\boldsymbol{x}) = -\frac{q}{3!}\, \frac{d^3}{dt^3}\, |\boldsymbol{x}-\boldsymbol{z}(t)|^2, \quad \boldsymbol{A}_{\mathrm{rr}}(t,\boldsymbol{x}) = -q\frac{d}{dt}\, \boldsymbol{v}(t).$$

Evaluating the time derivatives gives

$$\Phi_{\mathrm{rr}}(t,\boldsymbol{x}) = \frac{q}{3}\, (\boldsymbol{x}-\boldsymbol{z})\cdot\dot{\boldsymbol{a}} - q\boldsymbol{v}\cdot\boldsymbol{a}, \quad \boldsymbol{A}_{\mathrm{rr}}(t,\boldsymbol{x}) = -q\boldsymbol{a},$$

where $\dot{\boldsymbol{a}} = d\boldsymbol{a}/dt$. Substituting these expressions into Eqs. (2.18), we obtain the radiation-reaction fields,

$$\boldsymbol{E}_{\mathrm{rr}} = \frac{2}{3}\, q\dot{\boldsymbol{a}}, \qquad \boldsymbol{B}_{\mathrm{rr}} = 0. \tag{3.5}$$

Finally, Eq. (2.19) gives us the radiation-reaction force:

$$\boldsymbol{F}_{\mathrm{rr}} = \frac{2}{3}\, q^2\dot{\boldsymbol{a}}. \tag{3.6}$$

The equation of motion of a charged particle is therefore $m\boldsymbol{a} = \boldsymbol{F}_{\mathrm{ext}} + \frac{2}{3}q^2\dot{\boldsymbol{a}}$, in which the first term represents an externally applied force.

We may now substantiate our previous claim that the radiation-reaction force is associated with a loss of energy in the source. The rate at which the force is doing work is given by

$$\dot{W} = \boldsymbol{F}_{\mathrm{rr}}\cdot\boldsymbol{v} = \frac{2}{3}\, q^2\dot{\boldsymbol{a}}\cdot\boldsymbol{v} = \frac{2}{3}\, q^2\left[\frac{d}{dt}(\boldsymbol{a}\cdot\boldsymbol{v}) - \boldsymbol{a}^2\right].$$

Upon averaging, assuming either that the motion is periodic or unaccelerated at early and late times, we obtain

$$\langle \dot{W} \rangle = -\frac{2}{3}\, q^2 \boldsymbol{a}^2. \tag{3.7}$$

The quantity appearing on the right-hand side is (minus) the energy radiated per unit time by the charged particle. We therefore have energy balance, on the average.

Equation (3.6) is the leading term in an expansion of the radiation-reaction force in powers of the particle's velocity. A careful examination of the higher-order terms in Eq. (3.4) reveals that in fact, Eq. (3.6) is *exact* in the particle's MCLF [1]. To generalize to other Lorentz frames, we must find the unique tensorial equation that reduces to Eq. (3.6) in the MCLF. It is easy to check that the correct answer is

$$F^\alpha_{\rm rr} = \frac{2}{3}\, q^2 \bigl(\delta^\alpha_{\ \beta} + u^\alpha u_\beta\bigr) \dot{a}^\beta, \tag{3.8}$$

where $\dot{a}^\alpha(\tau) = da^\alpha/d\tau$. The equations of motion for a point particle of charge q and mass m are therefore

$$m a^\alpha = F^\alpha_{\rm ext} + \frac{2}{3}\, q^2 \bigl(\delta^\alpha_{\ \beta} + u^\alpha u_\beta\bigr) \dot{a}^\beta,$$

where $F^\alpha_{\rm ext}$ is the external force. This is the *Lorentz-Dirac equation*. In the following sections we will derive it using fully covariant methods.

Problem 4. Show that Eq. (3.8) reduces to Eq. (3.7) in the momentarily comoving Lorentz frame. Show also that the radiation-reaction force can be expressed as $F^\alpha_{\rm rr} = \frac{2}{3} q^2 (\dot{a}^\alpha - a^2 u^\alpha)$, where $a^2 = a_\alpha a^\alpha$.

IV. LIGHT-CONE MAPPING

The fully relativistic calculation of the potential and field associated with a point charge relies heavily on the causal structure of the retarded Green's function $G_{\rm ret}(x, x')$, which has support on the past light cone of the field point x. The field at x therefore depends on the state of the charge's motion at the instant at which its world line intersects the past light cone. Thus, the past light cone defines a natural mapping between the field-point x and a specific point z on the world line. In this section we develop the mathematical tools associated with this mapping. The results derived here will be used throughout the following sections.

Let x be a field point, and $z(u)$ the point at which the world line intersects the past light cone of x (see Fig. 1). Here, $u(x)$ denotes the specific value of the particle's proper time τ which corresponds to the intersection point; we shall refer to u as the *retarded time*. Clearly, u is determined by x; it is obtained by solving the equation

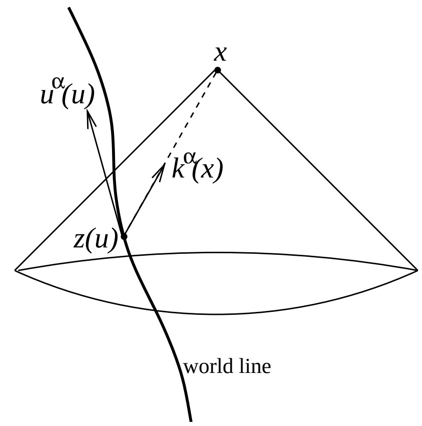

FIG. 1. Light-cone mapping between an arbitrary field point x and the point $z(u)$ on the world line.

$$0 = \sigma(x, u) \equiv \frac{1}{2}\, \eta_{\alpha\beta} \bigl[x^\alpha - z^\alpha(u)\bigr] \bigl[x^\beta - z^\beta(u)\bigr], \tag{4.1}$$

which states that the points x and $z(u)$ are linked by a null geodesic.

We will need an invariant measure of the distance between x and $z(u)$. Consider the scalar quantity

$$r(x) = -\eta_{\alpha\beta} \bigl[x^\alpha - z^\alpha(u)\bigr]\, u^\beta(u). \tag{4.2}$$

In the MCLF associated with the particle's motion at the retarded time, we have $r = t - z^0(u)$, which is just the time required for light to propagate from $z(u)$ to x. Because the speed of light is set to unity, r is also the spatial distance between these two points. Thus, the invariant $r(x)$ is the distance between x and $z(u)$, as measured in the MCLF; we may refer to it as the *retarded distance* between the field point and the particle. Notice that because u is determined by x, there is no need to make the dependence on u explicit in r: we write $r(x)$ and not $r(x, u)$.

The vector $x^\alpha - z^\alpha(u)$ is a null vector pointing from $z(u)$ to x. It is useful to rescale this vector by a factor r^{-1}, so as to define a new vector,

$$k^\alpha(x) = \frac{1}{r} \bigl[x^\alpha - z^\alpha(u)\bigr]. \tag{4.3}$$

By virtue of Eqs. (4.1) and (4.2), this satisfies

$$k_\alpha(x) k^\alpha(x) = 0, \qquad k_\alpha(x) u^\alpha(u) = -1; \tag{4.4}$$

the second relation provides a convenient normalization for the null vector.

Because x and $z(u)$ are linked by the light-cone mapping, a change of field point x generally comes with a change in u. (An exception arises if the displacement is directed along the null geodesic linking x to z.) For infinitesimal displacements, it is easy to relate a variation in x to the corresponding change in u. Suppose that x

is displaced to the new field point $x + \delta x$. The new intersection point in then $z(u + \delta u)$, and these points are related by Eq. (4.1), $\sigma(x + \delta x, u + \delta u) = 0$. Expanding this to first order in δx and δu, and using Eqs. (4.3) and (4.4), we obtain $k_\alpha \delta x^\alpha + \delta u = 0$, or

$$\frac{\partial u}{\partial x^\alpha} = -k_\alpha. \tag{4.5}$$

This result allows us to efficiently differentiate a function of x which contains an implicit reference to the point u. Let $f(x)$ be such a function. We may make the dependence on u explicit by writing $f(x) = F(x, u)$. Upon differentiation, we momentarily treat x and u as independent variables, and write $df = (\partial F/\partial x^\alpha)\, dx^\alpha + (\partial F/\partial u)\, du$. We then use Eq. (4.5) to obtain the full derivative with respect to x^α:

$$\frac{\partial f}{\partial x^\alpha} = \left(\frac{\partial F}{\partial x^\alpha}\right)_u - k_\alpha \left(\frac{\partial F}{\partial u}\right)_x. \tag{4.6}$$

This is the differentiation rule under the light-cone mapping.

As a specific application of Eq. (4.6), it is easy to check that the derivative of the retarded distance is given by

$$r_\alpha \equiv r_{,\alpha} = -u_\alpha + (1 + r a_k) k_\alpha, \tag{4.7}$$

where $a_k = a_\alpha k^\alpha$ is the component of the acceleration $a^\alpha = du^\alpha/d\tau$ in the direction of k^α. It is understood that in Eq. (4.7), all world-line quantities (such as u^α and a^α) are to be evaluated at the retarded time u. Notice that Eq. (4.7) implies

$$k_\alpha(x) r^\alpha(x) = 1. \tag{4.8}$$

As another example, we may use Eq. (4.6) to calculate the derivative of k^α. The result is

$$k_{\alpha,\beta} = \frac{1}{r}\Big(g_{\alpha\beta} + k_\alpha u_\beta + u_\alpha k_\beta - k_\alpha k_\beta\Big) - a_k k_\alpha k_\beta. \tag{4.9}$$

This implies that k^α satisfies the geodesic equation, $k_{\alpha,\beta} k^\beta = 0$. We also note that $k^\alpha_{\ ,\alpha} = 2/r$.

Problem 5. Verify Eqs. (4.7) and (4.9). Show that r is an affine parameter on the null geodesic linking x to $z(u)$.

V. ELECTROMAGNETIC FIELD OF A RELATIVISTIC POINT CHARGE

The retarded solution to Eq. (2.4), with the current density given by Eq. (2.1), is $A^\alpha(x) = q \int d\tau\, u^\alpha G_{\rm ret}(x, z)$, in which both z^α and u^α are functions of proper time τ; the integration is over the complete world line. (We again discard the possibility of adding a solution to the homogeneous equation, which would give rise to an external field $F^{\alpha\beta}_{\rm ext}$.) Substituting Eq. (2.6), we obtain

$$A^\alpha(x) = q \int d\tau\, u^\alpha \theta(t - z^0) \delta(\sigma),$$

where $\sigma(x, z)$ is defined by Eq. (2.8). The δ-function selects two points on the world line, those which satisfy the relation $\sigma(x, z) = 0$. One solution gives the retarded time u, defined by the condition that $z(u)$ causally precedes x; the other gives the advanced time v, defined by the condition that x precedes $z(v)$. The θ-function rejects the advanced-time solution, so that $A^\alpha(x)$ depends on the state of the particle's motion at the retarded time only.

The integral can be evaluated by changing the variable of integration to σ, writing $d\tau = d\sigma/\dot\sigma$. We note that since σ goes from negative to positive values as τ passes through the retarded time u, $\dot\sigma$ is positive at $\tau = u$. In fact, a quick calculation using Eqs. (4.1) and (4.2) reveals that $\dot\sigma(\tau = u) = r(x)$. The vector potential is therefore given by

$$A^\alpha(x) = q \frac{u^\alpha(u)}{r(x)}. \tag{5.1}$$

This is the well-known *Liénard-Wichert* potential.

The electromagnetic field $F_{\alpha\beta}$ is obtained from the vector potential by using the differentiation rule of Eq. (4.6). Using $u_{\alpha,\beta} = -a_\alpha k_\beta$ and Eq. (4.7), we obtain

$$F_{\alpha\beta} = \frac{2q}{r}\Big(a_{[\alpha} k_{\beta]} + a_k\, u_{[\alpha} k_{\beta]}\Big) + \frac{2q}{r^2}\, u_{[\alpha} k_{\beta]}, \tag{5.2}$$

where we have suppressed the dependence of the world-line quantities (such as u^α, a^α, and $a_k = a_\alpha k^\alpha$) on the retarded time u. The square brackets denote antisymmetrization of the indices: $a_{[\alpha} k_{\beta]} = \frac{1}{2}(a_\alpha k_\beta - k_\alpha a_\beta)$. (Below, we will use round brackets to denote symmetrization of the indices.) It is interesting to note that the part of the electromagnetic field which scales as r^{-1} is proportional to the particle's acceleration; this is the *radiative* part of the field. On the other hand, the part which scales as r^{-2} does not involve the acceleration; this is the *bound* — or *Coulomb* — part of the field.

Problem 6. Check the validity of Eq. (5.2). Verify that the electromagnetic field satisfies the vacuum Maxwell equations, $F^{\alpha\beta}_{\ \ ,\beta} = 0$, away from the world line.

It is straightforward to substitute Eq. (5.2) into Eq. (2.13) to calculate the electromagnetic field's stress-energy tensor. This computation reveals a natural decomposition into radiative and bound components,

$$T^{\alpha\beta}_{\rm em} = T^{\alpha\beta}_{\rm rad} + T^{\alpha\beta}_{\rm bnd}, \tag{5.3}$$

where

$$T^{\alpha\beta}_{\rm rad} = \frac{q^2}{4\pi r^2} \Bigl(a^2 - {a_k}^2 \Bigr) k^\alpha k^\beta \qquad (5.4)$$

is the radiative component, and

$$\begin{aligned} T^{\alpha\beta}_{\rm bnd} = \frac{q^2}{2\pi r^3} \biggl[k^{(\alpha} a^{\beta)} + a_k \Bigl(k^{(\alpha} u^{\beta)} - k^\alpha k^\beta \Bigr) \biggr] \\ + \frac{q^2}{4\pi r^4} \biggl[2k^{(\alpha} u^{\beta)} - k^\alpha k^\beta + \frac{1}{2}\, \eta_{\alpha\beta} \biggr] \end{aligned} \qquad (5.5)$$

is the bound — or Coulomb — component; we use the notation $a^2 = a_\alpha a^\alpha$. This decomposition is meaningful because each component is separately conserved away from the world line,

$$\partial_\beta T^{\alpha\beta}_{\rm rad} = 0, \qquad \partial_\beta T^{\alpha\beta}_{\rm bnd} = 0, \qquad (r \neq 0). \qquad (5.6)$$

Furthermore, the interpretation of $T^{\alpha\beta}_{\rm rad}$ as the radiative part of the stress-energy tensor is motivated by the fact that it scales as r^{-2} and is proportional to $k^\alpha k^\beta$.

Problem 7. Check the validity of Eqs. (5.3)–(5.5). Verify that $T^{\alpha\beta}_{\rm rad}$ is separately conserved away from the world line.

VI. THE RADIATION-REACTION FORCE

As we have seen in Sec. III, the radiation-reaction force can be calculated on the basis of a radiation-reaction potential equal to half the difference between the retarded and advanced potentials. The retarded potential was calculated in Sec. V, and given in Eq. (5.1). A similar calculation reveals that the advanced potential is

$$A^\alpha_{\rm adv}(x) = q \frac{u^\alpha(v)}{r_{\rm adv}(x)}, \qquad (6.1)$$

where v is the advanced time, determined by the relation $\sigma(x,z) = 0$ and the condition that x precedes $z(v)$, and

$$r_{\rm adv}(x) = -\eta_{\alpha\beta} \bigl[z^\alpha(v) - x^\alpha \bigr]\, u^\beta(v) \qquad (6.2)$$

is the advanced distance.

To construct the radiation-reaction potential, it is advantageous to express $A^\alpha_{\rm adv}(x)$ in terms of retarded quantities. Because we are interested in a field point x very close to the world line, this transcription is easily carried out by Taylor expansion. We shall therefore expand $\Delta\tau \equiv v - u$ and $r_{\rm adv}(x)$ in powers of the retarded distance r. It is easy to see that formally, $\Delta\tau$ and r are of the same order of smallness.

We begin by substituting the expansion

$$\begin{aligned} z^\alpha(v) = z^\alpha + u^\alpha\, \Delta\tau + \frac{1}{2}\, a^\alpha\, \Delta\tau^2 + \frac{1}{6}\, \dot{a}^\alpha\, \Delta\tau^3 \\ + \frac{1}{24}\, \ddot{a}^\alpha\, \Delta\tau^4 + O(\Delta\tau^5), \end{aligned}$$

in which all quantities on the right-hand side are evaluated at the retarded time u, into the relation $\sigma(x, z(v)) = 0$. After some algebra, using the relations $\sigma(x, z(u)) = 0$ and $x^\alpha - z^\alpha(u) = r k^\alpha$, we obtain

$$\begin{aligned} 0 = 2r\, \Delta\tau - (1 + r a_k)\, \Delta\tau^2 - \frac{1}{3}\, r \dot{a}_k\, \Delta\tau^3 \\ - \frac{1}{12} \bigl(r \ddot{a}_k + a^2 \bigr)\, \Delta\tau^4 + O(\Delta\tau^5), \end{aligned}$$

where, for example, $\dot{a}_k = \dot{a}_\alpha k^\alpha$. This relation implies that to leading order, $\Delta\tau = 2r$ — the time delay is twice the spatial distance between the world line and the field point. Calculating higher-order terms is straightforward, if slightly tedious. The result is

$$\Delta\tau = 2r \biggl[1 - a_k r + \biggl({a_k}^2 - \frac{1}{3}\, a^2 - \frac{2}{3}\, \dot{a}_k \biggr) r^2 + O(r^3) \biggr]. \qquad (6.3)$$

We recall that $\Delta\tau$ stands for $v - u$, and that all quantities on the right-hand side refer to the retarded time u.

To express $r_{\rm adv}$ in terms of r, we substitute the previous expansion for $z^\alpha(v)$ and a similar expansion for $u^\alpha(v)$,

$$u^\alpha(v) = u^\alpha + a^\alpha\, \Delta\tau + \frac{1}{2}\, \dot{a}^\alpha\, \Delta\tau^2 + \frac{1}{6}\, \ddot{a}^\alpha\, \Delta\tau^3 + O(\Delta\tau^4),$$

into Eq. (6.2). (Here it is sufficient to keep terms only up to order $\Delta\tau^3$.) After using Eq. (6.3), we obtain

$$r_{\rm adv} = r + \frac{2}{3} \bigl(a^2 + \dot{a}_k \bigr) r^3 + O(r^4). \qquad (6.4)$$

We may also use Eq. (6.3) to express $u^\alpha(v)$ as an expansion in powers of r:

$$u^\alpha(v) = u^\alpha + 2a^\alpha\, r + 2 \bigl(\dot{a}^\alpha - a_k a^\alpha \bigr)\, r^2 + O(r^3). \qquad (6.5)$$

Again, all quantities on the right-hand side refer to the retarded time u.

Problem 8. Check the validity of Eqs. (6.3)–(6.5). Give a physical interpretation to the fact that $r_{\rm adv} = r$ if the motion is unaccelerated.

The advanced potential is obtained by substituting Eqs. (6.4) and (6.5) into Eq. (6.1). The result is

$$\begin{aligned} A^\alpha_{\rm adv}(x) = q \frac{u^\alpha}{r} + 2q a^\alpha + 2q \biggl[\dot{a}^\alpha - a_k a^\alpha \\ - \frac{1}{3} \Bigl(a^2 + \dot{a}_k \Bigr) u^\alpha \biggr] r + O(r^2). \end{aligned}$$

The radiation-reaction potential is therefore

$$\begin{aligned} A^\alpha_{\rm rr}(x) &= \frac{1}{2}\left[A^\alpha_{\rm ret}(x) - A^\alpha_{\rm adv}(x)\right] \\ &= -qa^\alpha - q\Big[\dot a^\alpha - a_k a^\alpha - \frac{1}{3}\Big(a^2 + \dot a_k\Big)u^\alpha\Big]r \\ &\quad + O(r^2). \end{aligned} \tag{6.6}$$

The terms of order r^2 and higher will not contribute to the radiation-reaction force, because after differentiation they give rise to terms which vanish on the world line.

Differentiation of the radiation-reaction potential proceeds with the help of Eq. (4.6), using such results as Eq. (4.7), (4.9), and $a_{\alpha,\beta} = -\dot a_\alpha k_\beta$. The calculation is straightforward, although it involves a fair amount of algebra. Many of the terms drop out after antisymmetrization of the indices, and we obtain

$$F^{\rm rr}_{\alpha\beta}(z) = -\frac{2}{3}\,q\big(\dot a_\alpha u_\beta - u_\alpha \dot a_\beta\big). \tag{6.7}$$

Substituting this into the Lorentz-force equation yields

$$F^\alpha_{\rm rr} = qF^{\alpha}_{{\rm rr}\ \beta}u^\beta = \frac{2}{3}\,q^2\big(\delta^\alpha_{\ \beta} + u^\alpha u_\beta\big)\dot a^\beta.$$

The equations of motion of a charged particle are therefore

$$ma^\alpha = F^\alpha_{\rm ext} + \frac{2}{3}\,q^2\big(\delta^\alpha_{\ \beta} + u^\alpha u_\beta\big)\dot a^\beta, \tag{6.8}$$

where $F^\alpha_{\rm ext}$ is an externally applied force. Thus, the Lorentz-Dirac equation follows directly from the realization that the half-retarded minus half-advanced potential is responsible for the radiation reaction. The role of the remaining part of the retarded potential will be elucidated in Sec. VIII.

Problem 9. Check that Eq. (6.7) does indeed follow from the radiation-reaction potential of Eq. (6.6). Check that the Lorentz-Dirac equation is compatible with the identity $\eta_{\alpha\beta}u^\alpha a^\beta = 0$.

VII. RETARDED COORDINATES

Our purpose in this section is to develop the mathematical tools required in an alternative derivation of the Lorentz-Dirac equation, to be presented in the following section. We shall construct a coordinate system for flat spacetime based on the retarded time u and retarded distance r; these coordinates will be centered on the accelerated world line. A broad class of such coordinate systems was considered by Newman and Unti [7]; ours is a specific example.

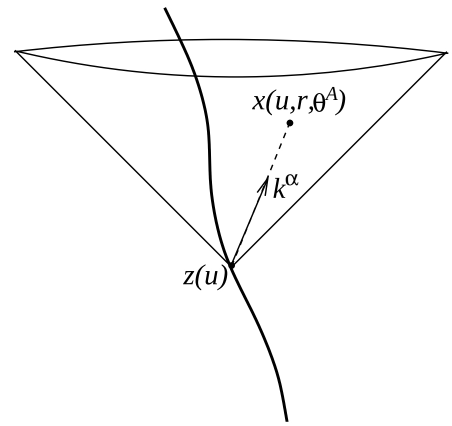

FIG. 2. A coordinate system centered on an accelerated world line.

The coordinate system (u, r, θ^A), where $\theta^A = (\theta, \phi)$ are two polar angles, is constructed as follows. We select the point $z(u)$ on the world line, and consider the forward light cone of this point (see Fig. 2). To all the spacetime events lying on the light cone we assign the same coordinate u. The null cone is generated by null rays emanating from $z(u)$ and radiating in all possible directions. We may choose a specific ray by selecting two angles, θ^A, which give the direction of the ray with respect to a reference axis. To all the spacetime events lying on this null ray we assign the same coordinates θ^A. Finally, a specific event on this null ray can be characterized by its affine-parameter distance from the apex; this is our fourth coordinate r.

The point $x^\alpha(u, r, \theta^A)$ in spacetime is therefore linked to the point $z^\alpha(u)$ on the world line by a null geodesic of affine-parameter length r; this null ray belongs to the light cone labelled by u and is further characterized by the angles θ^A specifying its direction on the cone. Let the vector $k^\alpha(u, \theta^A)$ be tangent to this null ray, which then admits the description $dx^\alpha/dr = k^\alpha$. Because k^α is a null vector, its normalization is arbitrary, and we are free to impose

$$\eta_{\alpha\beta}k^\alpha(u, \theta^A)u^\beta(u) = -1; \tag{7.1}$$

this also determines the normalization of the affine parameter r. Its interpretation as the retarded distance between x and $z(u)$ comes from the integrated form of the geodesic equation,

$$x^\alpha(u, r, \theta^A) = z^\alpha(u) + rk^\alpha(u, \theta^A), \tag{7.2}$$

which is identical to Eq. (4.3). Notice, however, the differences in point of view: In Sec. IV, k^α was considered to be a function of x; here, it is a function of u and θ^A which, together with r, *determine* the point x.

Equation (7.2) is the desired transformation between the inertial system x^α and the (noninertial) retarded coordinates (u, r, θ^A). So far, the construction is not unique: we have yet to decide how the angles θ^A are to be defined. In other words, we must give a prescription to locate the polar axis on each of the light cones; the angles θ^A will then be the polar angles associated with this axis.

We adopt the following prescription: We consider a Lorentz frame (t, x, y, z) which is momentarily comoving with the particle at the time $\tau = u$. In this frame, which we shall call the u-frame, $u^\alpha(u) \overset{*}{=} \delta^\alpha_t$, where "$\overset{*}{=}$" means "equals in the specified Lorentz frame". We orient the spatial axes so that the (purely spatial) acceleration vector points in the direction of the polar axis, $a^\alpha(u) \overset{*}{=} a(u)\,\delta^\alpha_z$, where $a(u)$ is the magnitude of the acceleration vector. Repeating this construction for every point on the world line gives us a polar axis on each of the null cones, and the angles θ^A will refer to this choice of axis. We let

$$k^\alpha(u, \theta^A) \overset{*}{=} (1, \sin\theta\,\cos\phi, \sin\theta\,\sin\phi, \cos\theta), \qquad (7.3)$$

which is compatible with the constraints coming from Eq. (7.1) and the fact that k^α is null.

Problem 10. Derive the following relations, which will be required in Sec. VII:

$$\frac{1}{4\pi}\int k^\alpha\,d\Omega = u^\alpha,$$

$$\frac{1}{4\pi}\int k^\alpha k^\beta\,d\Omega = \frac{1}{3}\,g^{\alpha\beta} + \frac{4}{3}\,u^\alpha u^\beta,$$

$$\frac{1}{4\pi}\int k^\alpha k^\beta k^\gamma\,d\Omega = \frac{1}{3}\left(u^\alpha g^{\beta\gamma} + u^\beta g^{\alpha\gamma} + u^\gamma g^{\alpha\beta}\right) + 2u^\alpha u^\beta u^\gamma,$$

where $d\Omega = \sin\theta\,d\theta d\phi$ is the element of solid angle. To reduce the work involved, express the null vector as $k^\alpha = u^\alpha + n^\alpha$, where n^α is a unit vector orthogonal to u^α. Rely on symmetry to argue that $\int n^\alpha\,d\Omega$ must vanish, and show that $\int n^\alpha n^\beta\,d\Omega$ must be proportional to $g^{\alpha\beta} + u^\alpha u^\beta$.

Equation (7.3) gives an explicit expression for $k^\alpha(u, \theta^A)$ in the u-frame. In the Lorentz frame (t', x', y', z') associated with the time u', defined so that $u^\alpha(u') \overset{*}{=} \delta^\alpha_{t'}$ and $a^\alpha(u') \overset{*}{=} a(u')\,\delta^\alpha_{z'}$, $k^\alpha(u', \theta^A)$ would take exactly the same form. This fact can be used to obtain a useful differential equation for $k^\alpha(u, \theta^A)$, which will allow us to work in a single Lorentz frame. To derive this, we take $u' = u + \delta u$ and work to first order in δu. The relative velocity between the two Lorentz frames is $u^\alpha(u+\delta u) - u^\alpha(u) = a^\alpha(u)\,\delta u$, which allows us to deduce the components of $u^\alpha(u+\delta u)$ in the u-frame: $u^\alpha(u+\delta u) \overset{*}{=} (1, 0, 0, a\,\delta u)$; we see that the boost parameter is $v = a(u)\,\delta u$. To first order in v, the Lorentz transformation between the two frames is given by $t' = t - vz$, $x' = x$, $y' = y$, and $z' = z - vt$. With this we can calculate the components of $k^\alpha(u+\delta u)$ in the u-frame: Using Eq. (7.3), we find $k^t(u+\delta u) \overset{*}{=} 1 + a(u)\cos\theta\,\delta u$, $k^x(u+\delta u) \overset{*}{=} \sin\theta\,\cos\phi$, $k^y(u+\delta u) \overset{*}{=} \sin\theta\,\sin\phi$, and $k^z(u+\delta u) \overset{*}{=} \cos\theta + a(u)\,\delta u$. These results imply

$$\frac{\partial k^\alpha}{\partial u} \overset{*}{=} (a\cos\theta, 0, 0, a) \overset{*}{=} (a_k\,u^t, 0, 0, a^z).$$

An equivalent tensorial equation is

$$\frac{\partial k^\alpha}{\partial u} = a_k\,u^\alpha + a^\alpha; \qquad (7.4)$$

it holds in an arbitrary Lorentz frame. With the initial data of Eq. (7.3), Eq. (7.4) allows us to find $k^\alpha(u, \theta)$ at all times.

Problem 11. Derive the relations $k_\alpha \partial k^\alpha/\partial u = 0$ and $u_\alpha \partial k^\alpha/\partial u + a_k = 0$, and show that Eq. (7.4) is compatible with them.

We now turn to the calculation of the metric in the coordinates (u, r, θ^A). Differentiation of Eq. (7.2) yields

$$dx^\alpha = \bar{u}^\alpha\,du + k^\alpha\,dr + e^\alpha_A\,d\theta^A, \qquad (7.5)$$

where

$$\bar{u}^\alpha = u^\alpha + r\frac{\partial k^\alpha}{\partial u} = (1 + ra_k)u^\alpha + ra^\alpha \qquad (7.6)$$

and

$$e^\alpha_A = r\frac{\partial k^\alpha}{\partial \theta^A}. \qquad (7.7)$$

The metric follows by substituting Eq. (7.5) into $ds^2 = \eta_{\alpha\beta}\,dx^\alpha dx^\beta$ and calculating all relevant inner products between $\bar{u}^\alpha$, k^α, and e^α_A. For example, the components of e^α_A in the u-frame can easily be obtained from Eq. (7.3), and we infer $k_\alpha e^\alpha_A = 0 = u_\alpha e^\alpha_A$, as well as

$$\eta_{\alpha\beta} e^\alpha_A e^\beta_B = \mathrm{diag}(r^2, r^2\sin^2\theta);$$

these results hold in any Lorentz frame. At the end of a straightforward computation, we obtain

$$ds^2 = -\left[(1 + ra_k)^2 - r^2a^2\right]du^2 - 2\,dudr + 2ra_A\,dud\theta^A + r^2\,d\Omega^2, \qquad (7.8)$$

where $a^2 = a_\alpha a^\alpha$, $a_k = a_\alpha k^\alpha$, $a_A = a_\alpha e^\alpha_A$, and $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$ is the metric on a unit two-sphere.

When viewed in the retarded coordinates, the metric of flat spacetime looks horribly complicated — there are many nondiagonal terms, and the metric components depend on the coordinates. What is then the particular advantage of these coordinates? The answer comes from Sec. V, in which we saw that the electromagnetic field of a charged particle is naturally expressed in terms of u and

r. This is true as well of the field's stress-energy tensor, which will be involved in the calculations of Sec. VIII. In fact, these calculations will not involve the metric directly. What we shall need, however, is an expression for dV, the four-dimensional volume element. In spite of the grim appearance of the metric, this is simple:

$$dV = \sqrt{-g}\, d^4x = r^2 dudrd\Omega, \tag{7.9}$$

where $d\Omega = \sin\theta\, d\theta d\phi$ is the element of solid angle. This result is established by computing g, the determinant of the matrix formed by the metric elements; the square root of $-g$ is the Jacobian of the coordinate transformation $x^\alpha(u, t, \theta^A)$.

Problem 12. Check the validity of Eqs. (7.8) and (7.9). Show that $\sqrt{-g}$ is the Jacobian of the coordinate transformation.

Another result we shall need is

$$d\Sigma_\alpha = r_\alpha\, r^2 dud\Omega, \tag{7.10}$$

which gives the (outward-directed) surface element of a three-cylinder $r =$ constant in Minkowski spacetime. Here, $r_\alpha = \partial r/\partial x^\alpha$ is the gradient of r in any coordinate system x^α. In the usual Lorentzian coordinates, r_α was worked out in Eq. (4.7). In the retarded coordinates, $r_\alpha = \delta^r{}_\alpha$.

The derivation of Eq. (7.10) proceeds as follows. The vectorial surface element on a hypersurface Σ is defined by $d\Sigma_\alpha = n_\alpha dA$, where n^α is the hypersurface's unit normal (pointing in the outward direction) and dA is the three-dimensional surface element. In our application, Σ is a surface of constant r whose induced metric is obtained by putting $dr = 0$ in Eq. (7.8):

$$ds^2_\Sigma = -\left[(1 + ra_k)^2 - r^2a^2\right] du^2 + 2ra_A\, dud\theta^A + r^2\, d\Omega^2.$$

The surface element is then given by $dA = \sqrt{-g_\Sigma}\, d^3x$, where g_Σ is the determinant of the induced metric. Because Σ is a surface of constant r, the unit normal must be proportional to the gradient of r. We therefore let $n_\alpha = \lambda r_\alpha$, and determine λ by making sure that n_α is properly normalized. This gives $\lambda^{-2} = g^{\alpha\beta} r_\alpha r_\beta$. In the retarded coordinates, $\lambda^{-2} = g^{rr}$, a component of the inverse metric. This is given by cofactor$(g_{rr})/g$, where the cofactor of a matrix element is the determinant obtained by eliminating the row and column to which this element belongs. It is easy to see that cofactor$(g_{rr}) = g_\Sigma$, so that $\lambda = \sqrt{g/g_\Sigma}$. Combining these results, we see that $d\Sigma_\alpha = n_\alpha\, dA$ reduces to $d\Sigma_\alpha = r_\alpha \sqrt{g}\, d^3x$, which is just Eq. (7.10).

Problem 13. A particle moving with uniform acceleration in the z direction has a world line described by $z^0 = a^{-1}\sinh(a\tau)$, $z^1 = z^2 = 0$, and $z^3 = a^{-1}\cosh(a\tau)$, where a, the magnitude of the acceleration vector, is a constant. These components of $z^\alpha(\tau)$ are given in the "laboratory frame", in which the particle is momentarily at rest at $t = 0$.

Calculate the components of $k^\alpha(u, \theta^A)$ in the laboratory frame. Show that the transformation to the retarded coordinates is given by

$$\begin{aligned} t &= a^{-1}\sinh\lambda + r(\cosh\lambda + \cos\theta\, \sinh\lambda), \\ x &= r\sin\theta\, \cos\phi, \\ y &= r\sin\theta\, \sin\phi, \\ z &= a^{-1}\cosh\lambda + r(\sinh\lambda + \cos\theta\, \cosh\lambda), \end{aligned}$$

where $\lambda = au$. Substitute this into the Minkowski metric, $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, and show that you agree with Eq. (7.8).

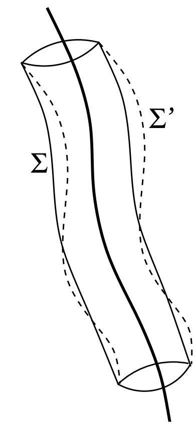

FIG. 3. The world tube Σ enclosing the world line. The dashed lines show a deformed world tube Σ'.

VIII. DIRAC'S DERIVATION

Dirac's derivation of the radiation-reaction force is based upon considerations of energy-momentum conservation. It goes as follows. We enclose the world line within a thin world tube Σ — a three-cylinder whose shape is *a priori* arbitrary (see Fig. 3) — and we calculate how much electromagnetic-field momentum flows across this surface per unit proper time. We then demand that this change in momentum be balanced by a corresponding change in the particle's momentum, so that the *total* momentum is properly conserved. Provided that the particle's momentum is correctly identified, this analysis yields the Lorentz-Dirac equation.

Quite generally, the flux of four-momentum across a hypersurface Σ is given by

$$\Delta P^\alpha = \int_\Sigma T^{\alpha\beta}\, d\Sigma_\beta, \tag{8.1}$$

where $T^{\alpha\beta}$ is any conserved stress-energy tensor and $d\Sigma_\alpha$ the outward-directed surface element on Σ. In our application, Σ will be the world tube of Fig 3. How does ΔP^α depend on the shape of the world tube? The answer is that it does not: the momentum flowing through a deformed tube Σ' is equal to the flux across the original tube Σ, provided that the tubes begin and terminate on the same two-surfaces. (The tubes must therefore have the same "caps"; we do not want Σ' to be longer than Σ, thereby allowing more momentum to go through.)

The proof of this statement is based on the spacetime formulation of Gauss' theorem and the fact that the stress-energy tensor is conserved. Let $\Delta P'^\alpha$ be the momentum flux across the deformed tube Σ'. Let $\mathcal{V}$ be the four-dimensional region between Σ' and Σ; this region's boundary is denoted $\partial\mathcal{V}$ and consists of the union of Σ and Σ'. By Gauss' theorem, we have

$$\begin{aligned}\Delta P'^\alpha - \Delta P^\alpha &= \int_{\partial\mathcal{V}} T^{\alpha\beta}\, d\Sigma_\beta \\ &= \int_{\mathcal{V}} T^{\alpha\beta}{}_{,\beta}\, dV \\ &= 0,\end{aligned}$$

which proves the assertion. Because the shape of the world tube is irrelevant, we shall make the simplest choice and take Σ to be a hypersurface of constant r. The surface element on this world tube is given by Eq. (7.10). We will assume that r is small, so that Σ lies in the immediate vicinity of the world line. It will not be necessary, however, to take the limit $r \to 0$.

We have seen in Sec. V that the stress-energy tensor of the electromagnetic field is naturally decomposed into radiative and bound parts which are separately conserved away from the world line. By virtue of Eqs. (4.7) and (5.4), the radial component of the radiative stress-energy tensor is given by

$$T^{\alpha\beta}_{\mathrm{rad}}\, r_\beta = \frac{q^2}{4\pi r^2}\Big(a^2 - {a_k}^2\Big)k^\alpha,$$

and the flux of radiative momentum is

$$\Delta P^\alpha_{\mathrm{rad}} = \frac{q^2}{4\pi}\int \big(a^2 - {a_k}^2\big)k^\alpha\, du d\Omega.$$

Notice that the factor r^2 in the surface element has canceled out the factor r^{-2} in the radiative stress-energy tensor; the result is independent of r. The rate of momentum change is

$$\frac{dP^\alpha_{\mathrm{rad}}}{du} = \frac{q^2}{4\pi}\int \big(a^2 - {a_k}^2\big)k^\alpha\, d\Omega.$$

The angular integration can be handled via the results of Problem 10, and we arrive at

$$\frac{dP^\alpha_{\mathrm{rad}}}{du} = \frac{2}{3}\, q^2 a^2 u^\alpha. \tag{8.2}$$

This is the amount of radiative momentum crossing a surface $r =$ constant per unit proper time. In the MCLF, this equation reduces to $dE/dt = \frac{2}{3}q^2\boldsymbol{a}^2$, the standard Larmor formula.

Going back to Eqs. (4.7) and (5.5), we find that the radial component of the bound stress-energy tensor is given by

$$T^{\alpha\beta}_{\mathrm{bnd}}\, r_\beta = \frac{q^2}{4\pi r^3}\left[a^\alpha + a_k\Big(u^\alpha - \frac{3}{2}k^\alpha\Big)\right] + \frac{q^2}{4\pi r^4}\big(u^\alpha - k^\alpha\big).$$

After integration we notice that the most singular terms, those which are proportional to r^{-4}, go away. The r^{-3} terms, however, give a nonzero result:

$$\frac{dP^\alpha_{\mathrm{bnd}}}{du} = \frac{q^2}{2r}\, a^\alpha. \tag{8.3}$$

This is the rate of change of the bound momentum. We shall have to interpret this result.

Problem 14. Check the validity of Eqs. (8.2) and (8.3).

The rate of change of electromagnetic momentum is given by the sum of Eqs. (8.2) and (8.3),

$$\frac{dP^\alpha_{\mathrm{em}}}{du} = m_{\mathrm{em}} a^\alpha + \frac{2}{3}\, q^2 a^2 u^\alpha, \tag{8.4}$$

where we have written $q^2/(2r) = m_{\mathrm{em}}$ in anticipation of our eventual interpretation of the first term. Dirac now postulates that the total momentum, $P^\alpha_{\mathrm{em}} + P^\alpha_{\mathrm{mech}}$, must be conserved:

$$\frac{dP^\alpha_{\mathrm{em}}}{du} + \frac{dP^\alpha_{\mathrm{mech}}}{du} = 0. \tag{8.5}$$

Here, P^α_{mech} is the mechanical momentum associated with the point particle itself, and Eq. (8.5) becomes an equation of motion if the correct relation between P^α_{mech} and the world-line quantities can be identified. This necessitates a second postulate.

It would be tempting to make the identification $P^\alpha_{\mathrm{mech}} = m_0 u^\alpha$, with m_0 describing the particle's material mass. Combining this with Eqs. (8.4) and (8.5), we would obtain $ma^\alpha = -\frac{2}{3}q^2a^2u^\alpha$, in which $m = m_0 + m_{\mathrm{em}}$ would be interpreted as the particle's physical mass. This, unfortunately, is a nonsensical result, because the acceleration cannot be proportional to the four-velocity.

We must therefore revisit our expression for the mechanical momentum, and allow it to be a more complicated function of the world-line quantities. We seek an expression of the form $P^\alpha_{\mathrm{mech}} = m_0 u^\alpha + cq^2 a^\alpha$, in which c is a dimensionless constant to be determined. (Other possibilities exist, but as Dirac says, "they are much more complicated than [this], so that one would hardly expect them to apply to a simple thing like an electron".) Combining this with Eqs. (8.4) and (8.5) yields $ma^\alpha = -q^2(c\dot{a}^\alpha + \frac{2}{3}a^2u^\alpha)$. Demanding that the right-hand side be orthogonal to u^α gives $c = -\frac{2}{3}$.

Dirac's second postulate is therefore that the mechanical momentum must have the form

$$P^\alpha_{\rm mech} = m_0 u^\alpha - \frac{2}{3}\, q^2 a^\alpha, \tag{8.6}$$

with m_0 representing the purely material contribution to the particle's mass. This combines with the electromagnetic contribution $m_{\rm em} = q^2/(2r)$ — the electrostatic self-energy — to form the particle's physical mass m:

$$m = m_0 + m_{\rm em} = m_0 + \frac{q^2}{2r}. \tag{8.7}$$

Combining Eqs. (8.4)–(8.7) finally yields the Lorentz-Dirac equation,

$$ma^\alpha = \frac{2}{3}\, q^2 \bigl(\dot{a}^\alpha - a^2 u^\alpha\bigr). \tag{8.8}$$

We have seen in Sec. VI that the right-hand side of this equation is produced by the half-retarded minus half-advanced potential, which is regular on the world line. The role of the (singular) half-retarded plus half-advanced potential is now clear: it serves to renormalize the mass from m_0 to m; the difference is $m_{\rm em}$, the electrostatic self-energy of the particle.

It should be noted that while this derivation of the Lorentz-Dirac equation stays quite close to spirit of Dirac's own derivation, it differs from it in its technical aspects. Dirac employs a different world-tube construction, based on spacelike geodesics (as opposed to null geodesics) emanating from the world line. As a result, his calculations are much more involved than the ones presented here. Another consequence is that Dirac's expression for the mechanical momentum differs from Eq. (8.6); it is the expected $m_0 u^\alpha$. While Eq. (8.6) seems *ad hoc* and strange, it is precisely the expression that results from a careful consideration of the "caps" in Fig. 3 — it is the mechanical momentum that is appropriate for our particular world-tube construction. The proof of this statement relies heavily on distribution theory, and is presented in Ref. [3].

The final form of the Lorentz-Dirac equation is obtained by using the identity $a^2 = -\dot{a}_\alpha u^\alpha$ and inserting an external-force term on the right-hand side of Eq. (8.8). This gives

$$ma^\alpha = F^\alpha_{\rm ext} + \frac{2}{3}\, q^2 \bigl(\delta^\alpha_{\ \beta} + u^\alpha u_\beta\bigr)\dot{a}^\beta. \tag{8.9}$$

If the external force is produced by an external electromagnetic field $F^{\alpha\beta}_{\rm ext}$, then $F^\alpha_{\rm ext} = qF^{\alpha\beta}_{\rm ext}u_\beta$.

IX. DIFFICULTIES OF THE LORENTZ-DIRAC EQUATION

A surprising feature of the Lorentz-Dirac equation is that it involves the derivative of the acceleration vector. The equations of motion are therefore third-order differential equations for $z^\alpha(\tau)$, a very unusual situation that necessitates some reflection. For example, an issue that must be addressed is the specification of initial data for this third-order equation. In the usual case of second-order differential equations, the initial data consists of the particle's position and velocity at $\tau = 0$, and this information is sufficient to provide a unique solution. A third-order equation requires more, however, and it is not clear *a priori* what the additional piece of initial data should be.

To analyze this problem, let us consider the nonrelativistic limit of the Lorentz-Dirac equation, which we write in the form

$$\boldsymbol{a} - t_0 \dot{\boldsymbol{a}} = \frac{1}{m}\, \boldsymbol{F}_{\rm ext}, \tag{9.1}$$

where $\boldsymbol{F}_{\rm ext}$ is an external force and

$$t_0 = \frac{2}{3}\, \frac{q^2}{m} \tag{9.2}$$

is a constant with the dimension of time. For the purpose of this discussion we take $\boldsymbol{F}_{\rm ext}$ to be a given function of time. It is easy to check that the general solution to Eq. (9.1) is

$$\boldsymbol{a}(t) = e^{t/t_0}\biggl[\boldsymbol{b} - \frac{1}{mt_0}\int_{-\infty}^t \boldsymbol{F}_{\rm ext}(t')e^{-t'/t_0}\, dt'\biggr],$$

where we assume that $\boldsymbol{F}_{\rm ext}(t)$ goes to zero in the infinite past, sufficiently rapidly that the integral is well defined. The constant vector $\boldsymbol{b}$ is not constrained *a priori*; this is the third piece of data that must be specified to completely determine the motion of the particle.

To see how $\boldsymbol{b}$ should be chosen, let us specialize to a particularly simple case, in which the external force is turned on abruptly at $t = 0$ and stays constant thereafter: $\boldsymbol{F}_{\rm ext}(t) = \boldsymbol{f}\theta(t)$, where $\boldsymbol{f}$ is a constant vector. In this case we have

$$\boldsymbol{a}(t) = e^{t/t_0}\biggl[\boldsymbol{b} - \frac{\boldsymbol{f}}{m}\Bigl(1 - e^{-t/t_0}\Bigr)\theta(t)\biggr],$$

and we see that for arbitrary choices of $\boldsymbol{b}$, $\boldsymbol{a} \sim e^{t/t_0}$ for $t \gg t_0$. Even though the applied force is constant, the acceleration grows exponentially with time. This is the problem of *runaway solutions*, which occurs also in the general case. We notice, however, that these unphysical solutions can be eliminated if we set $\boldsymbol{b} = \boldsymbol{f}/m$. Then we find

$$\boldsymbol{a}(t) = \frac{\boldsymbol{f}}{m}\Bigl[\theta(-t)e^{t/t_0} + \theta(t)\Bigr].$$

Although $\boldsymbol{a}$ is now sensibly behaved for $t > 0$, we see that its behaviour is rather strange for $t < 0$: At a time $\sim t_0$ *prior* to the time at which the external force switches on, the acceleration begins to increase. This is the problem of *preacceleration*, which occurs also in the general case.

Problem 15. Show that the only non-runaway solution to Eq. (9.1) is

$$\boldsymbol{a}(t) = \frac{1}{m} \int_0^\infty \boldsymbol{F}_{\rm ext}(t + st_0)\, e^{-s}\, ds.$$

This solution displays preacceleration, because the acceleration at a time t depends on the external force acting at later times.

The problems of runaways and preacceleration cast a serious doubt on the validity of the Lorentz-Dirac equation. The root of the problem resides with the fact that we are trying to describe the motion of a point particle within a purely classical theory of electromagnetism. This cannot be done consistently. Indeed, a point particle cannot be taken too literally in a classical context; it must always be considered as an *approximation* to a non-singular, and extended, charge distribution. Essentially, the difficulties of the Lorentz-Dirac equation come from a neglect to take this observation into account.

Any extended charge distribution can be characterized by its total charge q and a number of higher multipole moments. For distances r that are large compared with the body's averaged radius ℓ, the electromagnetic field is well approximated by the monopole term, q/r^2, and in such circumstances, the point-particle description is appropriate. For smaller distances, however, this description loses its usefulness. The point-particle description is therefore an approximation in which the internal structure of the charge distribution can be considered to be irrelevant. This approximation is limited to distances $r \gg \ell$, and the "world line" of a point charge should properly be thought of as an extended world tube whose dimensions are nevertheless smaller than any other relevant scale. Such idealizations are common in physics; in hydrodynamics, for example, fluid elements are idealized as points.

We should therefore examine the conditions under which Eq. (9.1) is compatible with the restrictions associated with a point-particle description. Finite-size corrections can be incorporated into the Lorentz-Dirac equation, which becomes

$$\boldsymbol{a} = \frac{1}{m} \boldsymbol{F}_{\rm ext} + t_0 \dot{\boldsymbol{a}} + O(t_0 \ddot{a} \ell). \tag{9.3}$$

This reveals that the original form of the equation will be accurate, and the point-particle description valid, if $\ddot{a}\ell \ll \dot{a}$. To see what this means, let $a_c \sim F_{\rm ext}/m$ be a characteristic value for the acceleration, and t_c a characteristic time scale over which the acceleration changes. Then $\dot{a} \sim a_c/t_c$, $\ddot{a} \sim a_c/{t_c}^2$, and the correction term will be small if

$$\ell \ll t_c. \tag{9.4}$$

Thus, a necessary condition for the validity of the point-particle description is that the time scale over which the acceleration changes must be much longer than the light-travel time across the charge distribution.

Another condition must be imposed. Any extended charge distribution possesses an electrostatic self-energy given by $m_{\rm em} \sim q^2/\ell$. This self-energy contributes to the total mass of the charged body: $m = m_0 + m_{\rm em}$, where m_0 is the purely material contribution to the mass. The classical theory will be consistent if $m_{\rm em}$ does not exceed m: m_0 must be positive. Our second condition is therefore $q^2/\ell \lesssim m$, or

$$t_0 \lesssim \ell. \tag{9.5}$$

Thus, the time constant t_0 must be smaller than the light-travel time across the charge distribution.

Combining Eqs. (9.4) and (9.5), we arrive at our most important criterion: The point-particle description will be valid if and only if

$$t_0 \ll t_c. \tag{9.6}$$

Thus, changes in the acceleration must occur over time scales that are long compared with t_0. This requirement is clearly violated by runaway and preacceleration solutions. These predictions therefore lie outside the domain of validity of Eq. (9.1).

How do we then go about solving the runaway and preacceleration problems? Before answering this, notice first that $\boldsymbol{F}_{\rm ext}/m \sim a_c$ is the leading term on the right-hand side of Eq. (9.3), that $t_0\dot{\boldsymbol{a}}$ is smaller by a factor of order (t_0/t_c), and that the remaining term is smaller still, by a factor of order $(t_0/t_c)^2$. The Lorentz-Dirac equation can therefore be written as

$$\boldsymbol{a} = \frac{1}{m} \boldsymbol{F}_{\rm ext} + t_0 \dot{\boldsymbol{a}} + O({t_0}^2/{t_c}^2). \tag{9.7}$$

Our strategy will be to seek a replacement for this equation. (This strategy is the one adopted by Landau and Lifshitz; for a thorough review of failed alternatives, see Ref. [6].) The new equation will be equivalent to Eq. (9.7), in the sense that it will differ from it only by terms of order $(t_0/t_c)^2$, but it will be free of difficulties. In other words, the new equation will be compatible with the restrictions associated with a point-particle description.

The replacement procedure is simple. We start with the leading-order version of Eq. (9.7),

$$\boldsymbol{a} = \frac{1}{m} \boldsymbol{F}_{\rm ext} + O(t_0/t_c),$$

and we differentiate. This gives

$$\dot{\boldsymbol{a}} = \frac{1}{m}\,\dot{\boldsymbol{F}}_{\rm ext} + O(t_0/{t_c}^2),$$

which we substitute back into Eq. (9.7). The result is

$$\boldsymbol{a} = \frac{1}{m}\,\boldsymbol{F}_{\rm ext} + \frac{t_0}{m}\,\dot{\boldsymbol{F}}_{\rm ext} + O({t_0}^2/{t_c}^2),$$

which has the same degree of accuracy as Eq. (9.7). But since this equation no longer involves $\dot{\boldsymbol{a}}$, it does not give rise to runaway and preacceleration solutions.

Our conclusion is therefore that the modified Lorentz-Dirac equation,

$$m\boldsymbol{a} = \boldsymbol{F}_{\rm ext} + t_0\dot{\boldsymbol{F}}_{\rm ext}, \tag{9.8}$$

is much better suited to govern the motion of a point charge, within the restrictions implied by the point-particle description. Equation (9.8) is formally equivalent to the original Lorentz-Dirac equation, in the sense that they are both accurate up to terms of order $(t_0/t_c)^2$. But because it is a second-order differential equation for $\boldsymbol{z}(t)$, Eq. (9.8) does not come with the difficulties associated with the original equation.

The method by which Eq. (9.7) was transformed into Eq. (9.8) is an application of a general technique known as *reduction of order*. This technique can also be applied to the relativistic equation,

$$a^\alpha = \frac{1}{m}\,F^\alpha_{\rm ext} + t_0\bigl(\delta^\alpha_{\ \beta} + u^\alpha u_\beta\bigr)\dot{a}^\beta.$$

To leading order, we have $a^\alpha = m^{-1}F^\alpha_{\rm ext}$, and differentiation yields $\dot{a}^\alpha = m^{-1}F^\alpha_{{\rm ext}\,,\beta}u^\beta$. Substituting this into the original equation, we obtain

$$ma^\alpha = F^\alpha_{\rm ext} + t_0\bigl(\delta^\alpha_{\ \beta} + u^\alpha u_\beta\bigr)F^\beta_{{\rm ext}\,,\gamma}u^\gamma, \tag{9.9}$$

the relativistic version of the modified Lorentz-Dirac equation.

Problem 16. Show that if the external force is provided by an external electromagnetic field $F^{\alpha\beta}_{\rm ext}$, then the modified Lorentz-Dirac equation takes the form

$$\begin{aligned} ma^\alpha = qF^\alpha_{{\rm ext}\ \beta}u^\beta + qt_0\biggl[&F^\alpha_{{\rm ext}\ \mu,\nu}u^\mu u^\nu \\ &+ \frac{q}{m}\bigl(\delta^\alpha_{\ \beta} + u^\alpha u_\beta\bigr)F^\beta_{{\rm ext}\ \mu}F^\mu_{{\rm ext}\ \nu}u^\nu\biggr]. \end{aligned}$$

ACKNOWLEDGMENTS

I am grateful to Eanna Flanagan for his comments on an earlier draft of this manuscript. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

---

[1] L.D. Landau and E.M. Lifshitz, *The classical theory of fields* (Pergamon, Oxford, 1962).
[2] P.A.M. Dirac, Proc. Roy. Soc. London **A167**, 148 (1938).
[3] C. Teitelboim, D. Villarroel, and C.G. van Weert, Riv. Nuovo Cimento **3**, 9 (1980).
[4] J.D. Jackson, *Classical electrodynamics* (Wiley, New York, 1975).
[5] F. Rohrlich, *Classical charged particles* (Addison-Wesley, Redwood City, 1990).
[6] E.E. Flanagan and R.M. Wald, Phys. Rev. D **54**, 6233 (1996).
[7] E. Newman and T. Unti, J. Math. Phys. **4**, 1467 (1963).