Missing Memristor Magnetic Flux Found.

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Abstract

The Strukov phenomenological model used to describe their memristor has been applied to memristors in general. Here we show that there is a critical error in the derivation of this model, specifically that the equation given for the speed of the boundary (between doped and undoped titanium dioxide) is incorrect. This leaves the derived expression for the memristance unsupported. In this paper we derive a new memristor theory based on magnetostatics and demonstrate its application to the Strukov (or HP), solution-processed TiO_2 and PEO-PANI (or organic) memristors. The magnetic flux expected from Chua's constitutive definition of the memristor is identified as the magnetic flux associated with the flow of oxygen vacancies within the material. The memristance as determined by the model is shown to be dependent on three spatial dimensions. This work allows us to combine the missing memristor magnetic flux and Chua's constitutive memristor equations with real-world device measurables for the first time.

1 Introduction

Memristors are, often nanoscale, electronic components that act as resistors with a memory. This combination of functionality and scalability has led to the suggestion that memristors are a potential route to increasing computational complexity in terms of Moore's Law [1]. Memristor theory has been successfully applied to describe the operation of synapses [2] and other components of neurons [3,4], as well as the processes of learning in snails [5] and amoeba [6]. Therefore, memristors may become vital components in attempts to create brain-like (neuromorphic) computers that are capable of learning (see for example [7] and [8]) and possibly higher-level functions, such as intelligence.

There are four fundamental circuit properties which describe a circuit's operation: the electrical potential difference, V, the electronic current, I, the magnetic flux, φ and the charge, q. Three pairs of relationships define the operation of the first three fundamental circuit elements: the resistor ($R = V/I$); the capacitor ($C = q/V$); the inductor ($I = \varphi/V$). A fourth was added in 1971, when Chua predicted the existence of a device that would relate q to φ, the memristor [9]. Because I and V are time differentials of q and φ, memristors produce distinctive non-linear I-V curves that have three important features [10]: (1) hysteresis (memory); (2) zero current at zero voltage; and (3) A.C. frequency dependence (the size of the hysteresis is related to frequency and it shrinks to nothing above critical frequency). The memristor concept was generalised by [2] to memristive systems. The memristor has only one state variable whereas a memristive system can be a function of more, and memristive systems have been used to model systems from across the sciences from an alternative circuit model of the neuron [2,3,11] to thermistors [12].

Between 1971 and 2008, no one who had read Chua's work was able to make a memristor and the idea languished in the drawer of theoretical curiosities. However, over this time period, many experimentalists had reported 'anomalous' I-V curves with variations on a pinched hysteresis loop, such as the first report of a memristor I-V curve in TiO_2 in 1968 [13], the creation of the PEO-PANI organic based memristive system [14,15] and many others, mostly Resistive Random Access Memory (ReRAM) devices [16–18]. Strukov et al were the first to finally unite the idea of the memristor with a practical example, by describing a working memristor [19] complete with a phenomenological model for its operation and references to Chua's theoretical work.

The Strukov memristor [19] (also often called the H.P. memristor after the owners of the work) consists of a layer of titanium dioxide of thickness D sandwiched between two platinum electrodes of width E and F as shown in figure 1. The titanium dioxide layer contains lattice defects caused by missing oxygen atoms; these oxygen vacancies act as n-type dopants and the distance they have drifted through the memristor is given by w, where $0 < w < D$. The resistivity of the stoichiometric, un-doped TiO_2 is higher than that of doped, non-stoichiometric $TiO_{(2-x)}$. Interconversion between the doped and un-doped forms, caused by the drift of oxygen vacancies, changes the total resistance, R, of the memristor and produces the pinched hysteresis loop in the I-V curve.

However, the Strukov model lacks a demonstrable magnetic flux term [19] and the magnetic flux of a memristor has not been experimentally measured. As Chua's memristor definition includes magnetic flux, this lack has led to questions as to whether the Strukov memristor is a Chua memristor [20]. In response, it has been suggested that the magnetic flux may be a theoretical construct and not related to a material property [16] and/or that it is the non-linear I-V relationship which defines the memristor [19], a definition which widens the field of memristors to include, for example, all ReRAM devices [16]. If there is a magnetic flux associated with the material system, we would be able to better apply Chua's theory to real systems. Which would allow us to understand the physics in a deeper way and, by creating a set of equations that relate device properties with memristive function, use this knowledge to design memristors applicable to our needs.

It will be shown here that there is a fundamental error in the derivation of Strukov's phenomenological model and that an alternative derivation of a memristor model for their device leads to a more coherent model of real-world memristor devices and a consolidating of Chua's definition with Strukov's device. As Strukov et al's model has been applied to other devices, and as many memristor devices are based on similar chemistry (e.g. other vacancy transporting semi-conductor memristors), we will demonstrate that the model in this paper also applies to other devices and anticipate that it can be extended to a wide range of memristor device.

This paper is structured in the following manner, first we will describe Strukov's system and then demonstrate where the error in the derivation arises and how this has led to erroneous ideas about the dimensionality and scale of memristors. We then formulate a different model based on standard magnetostatics. We then illustrate that it can satisfy Chua's theoretical formulation and includes magnetic flux. We will then describe how the model in general can be applied to other memristor systems.

2 The Strukov Model of the Strukov Memristor

2.1 Chua's Definition of the Memristor

There are four circuit measurables: V, I, q and φ. The definitions

$$q(t) \equiv \int_{-\infty}^{t} I(\tau) d\tau \tag{1}$$

and

$$\varphi(t) \equiv \int_{-\infty}^{t} V(\tau) d\tau \tag{2}$$

relate charge to current and voltage to flux. They also widen the description of charge from a quantity stored by a capacitor to total amount of charge that has passed through the circuit. Similarly the flux is time integral of the voltage applied over time rather than the quantity stored by an inductor. Thus these quantities are relevant to circuits without such devices in them.

Between four measurables, there are six pairs of interactions. Two are given by the definitions above, another three are the constitutive relations of the resistor, inductor and capacitor described in the introduction. Chua's ground-breaking contribution was to realize that there was a constitutive relationship missing, one which would define a device that relates charge to flux. He realised that this device would act in a passive, (i.e. non-storing) manner (like a resistor) and would give rise to a pinched hysteresis loop, and this hysteresis suggested the device would have memory, hence the name memristor, a contraction of memory-resistor. This constitutive relationship [9] he gave as:

$$M(q) \equiv \frac{d\varphi(q)}{dq} \tag{3}$$

where M is the memristance, a time-varying instantaneous resistance, which relates voltage, V, to current, I, in the following manner

$$V(t) = M(q(t))I(t)\,. \tag{4}$$

The time dependence of the memristance demonstrates that there will be a time or frequency based effect on the response of the device. If the voltage is varied too quickly for the device to respond the memristive behaviour collapses to ohmic conduction. Note that the time-variance is entirely due to the fact that q is a function of t, the memristance will not change with time if q does not change. These equations are for a charge controlled memristor, it is trivial to work out those for a flux controlled device. The equations reproduced in these sections are the definitions given in Chua's original theoretical work [9] (the description is expanded to memristive systems in later works [2]), Chua has never given a description of what the function M is or where it arises from.

2.2 Strukov's Derivation of the Strukov Memristor Model.

At the time of publication, Strukov's ground-breaking paper was the first example of a working memristor which excited much interest and founded a novel field and reignited interest in existing fields, such as ReRAM. Since then, as it was discovered that other memristor systems had been fabricated, the impact of that paper on memristor theory has been the usefulness of the

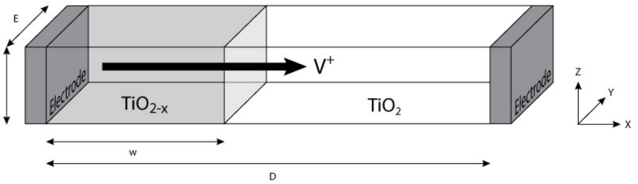

Figure 1: The Strukov memristor. The shaded area is doped low-resistance titanium dioxide, the unshaded area is stoichiometric high-resistance titanium dioxide. Vacancy, V^+, movement through the material is shown by the arrow. w marks the boundary between the two forms of titanium dioxide. The limits of the titanium dioxide layer in the y and z directions are E and F. As the vacancies move to the right along the x axis, the magnetic $\mathbf{B}$ field associated with them curls around in an anti-clockwise direction (not shown) and thus the surfaces that cross magnetic field lines are those in the x-y and x-z planes.

phenomenological model described within it. By providing a description for M, theorists were able to model memristor devices and make progress in memristor science, even whilst many of them did not have real devices to test their ideas on. The model was found to fit other memristor systems [21] and the modelling of the boundary has been improved (i.e. by using windowing functions and more realistic models for oxygen ion transport). However, there were two errors in the derivation, which has led to confusion over the magnetic flux and the claim (now experimentally proved false) that memristors had to be nanoscale [19]. In this section we shall go through the derivation from paper [19] in detail to highlight these issues and demonstrate why the derivation is incorrect.

2.2.1 The Details of Strukov's Derivation

To deal with the simplest possible case, they make two assumptions [19]:

1. ‘ohmic electronic conductance’, i.e. $V(t) = R(t)I(t)$, R may vary with time but at each instant in time, the current is wholly determined by the voltage and resistance;
2. ‘linear ionic drift in a uniform field with average ion mobility μ_v’.

Because the TiO_2 can inter-convert to $TiO_{(2-x)}$, with no volume change, the device can be modelled as two space-conserving resistors, one with resistance R_{off} and one with resistance R_{on}. The resistors are space-conserving because $TiO_{(2-x)}$ is made from the TiO_2 and vice-versa, thus they have made a zeroth assumption that matter is not created or destroyed in this system (not stated in the paper). Strukov et al started with the space-conserving variable resistors and gave the following formula for how the resistance changes as equation 5 in [19]

$$V(t) = \left(R_{\text{on}} \frac{w(t)}{D} + R_{\text{off}} \left(1 - \frac{w(t)}{D} \right) \right) I(t), \tag{5}$$

where $\frac{w(t)}{D}$ serves to give the fraction of the material which is the doped resistor and $\left(1 - \frac{w(t)}{D}\right)$ is the undoped resistor. They are using a simple model of two variable resistor with a runner between (see figure 2A in [19]), mathematically it is a proportional mixing of two resistors, which gives rise to a continuum change of resistance.

From equation 5 and assumption 1 (ohmic conduction) we can write an expression for the total resistance in the system, $R(t)$, as:

$$R(t) = R_{\text{on}} \frac{w(t)}{D} + R_{\text{off}} \left(1 - \frac{w(t)}{D} \right) . \tag{6}$$

Strukov et al concern themselves with the speed of the moving boundary, $\frac{dw(t)}{dt}$. This is modelled in relation to the average drift velocity of the oxygen dopants, $\mathbf{s}_v$. This arises from assumption 2: if we have linear ionic drift, then the ions move (on average) at the same speed across the entire device. Therefore because the boundary is the measure of the furthest reach of those ions, the boundary must move at the drift speed (because if the travelling front of ions moved (on average) faster or slower than the bulk average it would contradict the assertion that we have linear ionic drift). As the average drift speed is the scalar part of the average ionic drift velocity, we have

$$\frac{dw(t)}{dt} = |\mathbf{s}_v| . \tag{7}$$

Assumption 2 also states that the ions are drifting in a uniform electric field **L** with an average ionic mobility μ_v, which gives us (from writing out assumption 2, which is a definition of drift velocity):

$$\mathbf{s}_v = \mu_v \mathbf{L} \ . \tag{8}$$

A uniform electric field is given by the voltage across the titanium dioxide divided by the thickness of that material:

$$L = \frac{V}{D} \ . \tag{9}$$

So, they set $\frac{dw(t)}{dt} = |\mathbf{s}_v|$ and substitute equation 9 into 8 to get

$$\frac{dw(t)}{dt} = \mu_v \frac{V(t)}{D} \ , \tag{10}$$

which is not explicitly given in the paper [19] but is described in words in the derivation and arises naturally from the assumptions.

The authors actually report (as equation 6 in [19])

$$\frac{dw(t)}{dt} = \mu_v \frac{I R_{\text{on}}}{D} \ , \tag{11}$$

where assumption 1 (ohmic electronic conductance) has been used to substitute for V.

They then integrate both sides

$$\int \frac{dw(t)}{dt} dt = \int \mu_v \frac{I(t) R_{\text{on}}}{D} dt \ , \tag{12}$$

and make use of the definition of charge, given in equation 1 to get

$$w(t) = \mu_v \frac{R_{\text{on}} q(t)}{D} \ , \tag{13}$$

which is equation 7 in [19].

Equation 13 is then substituted into equation 5 to give

$$V(t) = \left(\frac{R_{\text{on}}^2 \mu_v q(t)}{D^2} + R_{\text{off}} - \frac{R_{\text{off}} R_{\text{on}} \mu_v q(t)}{D^2} \right) I(t) \ . \tag{14}$$

They compare equation 14 to Chua's constitutive relation given in equation 4 and conclude that $R(t)$ (the terms in the brackets) is the memristance, $M(q)$, where the time dependence of M arises entirely from $q(t)$.

They state that:

$$R_{\text{on}} \ll R_{\text{off}} \ , \tag{15}$$

which as $R_{\text{on}} \sim 1$ and $R_{\text{off}} \sim 160$ in their system, is not unreasonable. We shall thus take equation 15 as assumption 3. Therefore, as R_{on} is over 100 times smaller than R_{off} they drop the R_{on}^2 term (to make the equation simpler). Thus they report

$$M(q) = R_{\text{off}} \left(1 - \frac{\mu_v R_{\text{on}} q(t)}{D^2} \right) \ . \tag{16}$$

2.2.2 Consequences of the Strukov derivation

The derivation in [19] has been assumed to be correct and as a result of equation 16 several claims were made that have led to misunderstandings about memristors.

There is an error in the derivation associated with the system set-up, which is not a fatal flaw but an unrecognised assumption. Equation 6 is theoretically one-dimensional as it is dependent only on w. This arises from treating the memristor as two-space conserving resistors with a slider w to mix the relative amount of R_{off} and R_{on}. The model is also spatially one-dimensional because the amount of 3-D memristor material in the doped and un-doped state is approximated by only the proportion of the thickness D in that state. This is not a bad assumption given that (because we assume that the volume occupied by the memristive material does not change as a result of doping) the length and width of memristive material is the same for the doped and undoped memristive material. An alternative formulation would be to replace R_{on} and R_{off} terms with the resistivity of the two types of material (if we know it), and the volume that material occupies using the definition of resistivity.

From examination of Strukov's expression for M, the claim was made that memristance depends only on D [19], meaning that the model is spatially one-dimensional. However, M depends only on D and not any other dimensions of the device because no other spatial dimensions were included in the model at the start (for simplicity). Thus to conclude that M depends only on D is to make the error of drawing a conclusion that was an un-expressed assumption.

This error is relatively minor (indeed, no one other than the author of this paper seems to have noticed or remarked on it, see [22]) but the compound error associated with $\frac{1}{D^2}$ has had an effect. Strukov et al suggested that for appreciable memristance to be measured, D must be small, ideally nanoscale, to make $\frac{1}{D^2}$ (and thus the difference between the fully switched on and fully switched off resistances) large. Thus it was suggested that memristors could not be fabricated until nanoscale ($D \sim 10^{-9}$) film technology existed. There have since been several experiments that contradict this viewpoint [17,23,24]. However, the only reason that the memristance didn't depend on E or F was because they were purposely not included at the start. (This realisation does suggest the intriguing possibility of designing memristors with different properties based on their shape).

Another historical problem with equation 16 is the question of where the magnetic flux is. Because Chua's definition (see equation 3) included magnetic flux [9] it was expected that there should be an equation that related Strukov's memristance to a magnetic flux. Indeed, the authors state that the "magentic field does not play an explicit role in the mechanism of memristance" [19]. They concluded that memristance was just a theoretical concept and it was only the non-linear relation between the integrals of voltage and current that defined a memristor. Perhaps the lack of magnetic flux (and field) was an indication that the derivation was incorrect. As it happens, the quantity $\frac{D^2}{\mu_v}$ has the units of magnetic flux, but this quantity is not a magnetic flux. (Interestingly, if it were, the value of this quantity for the Strukov memristor would be so far outside of an expected value (see section 4.3) that it should have suggested that this equation was incorrect).

We will now go on to demonstrate that there is a critical error in the Strukov derivation which unfortunately requires that we forgo the use of this formulation of memristance.

2.3 Mathematical Disproof of the Strukov Derivation

The problem with the derivation arises from the substitution between 10 and 11 where IR is substituted for V.

2.3.1 Assumptions

For simplicity, we list the assumptions used by Strukov et al below:

- 0. Matter is conserved in this system
- 1. The system is instantaneously ohmic: $V(t) = R(t)I(t)$
- 2. The ions undergo linear ionic transport in a uniform field with an average ion mobility of μ_v
- 3. $R_{\text{on}} \ll R_{\text{off}}$

We also have other physical facts about the system, such as the value of the electron mobility, μ_e, that w varies between 0 and D, that all the resistances are more than zero, that D is non-zero and so on.

2.3.2 Problems with the resistance

Theorem 2.1. *The substitution of* R_{on} *into equation 10 that leads to equation 11 is incorrect*

Proof. Assumption 1 is $V(t) = R(t)I(t)$ where $R(t)$ is the total resistance and $I(t)$ is the total current. Therefore, when substituting for V in equation 11, the resulting equation should be

$$\frac{dw(t)}{dt} = \mu_v \frac{IR}{D} \; . \tag{17}$$

□

Theorem 2.2. *If equation 11 were correct, it leads to either a description of a non-memristor or a contradiction*

Proof. For equation 11 to be correct,

$$R = R_{\text{on}} \Rightarrow R_{\text{on}} \frac{w(t)}{D} + R_{\text{off}} \left(1 - \frac{w(t)}{D}\right) = R_{\text{on}} \; , \tag{18}$$

from substituting for R.

There are two ways this can be possible.

The first way is if $w(t) = D \; \forall \; t$, this implies that w can not be dependent on t i.e. it does not change,

$$\therefore \frac{dw(t)}{dt} = 0 \tag{19}$$

which, because

$$\mu_v, R, D \neq 0 \Rightarrow I(t) = 0 \; \forall \; t. \tag{20}$$

Equation 19 describes the system when it is stuck at the minimum resistance, which is equivalent to a resistor of resistance R_{on} and no longer fits the definition of the memristor. Equation 20 describes an un-powered device (that can never be turned on), thus it is also not a memristor.

The second way for $R = R_{\text{on}}$ is if $R_{\text{off}} = R_{\text{on}}$, then

$$R(t) = R_{\text{on}} \frac{w(t)}{D} + R_{\text{on}} \left(1 - \frac{w(t)}{D}\right) \; , \tag{21}$$

which if we rewrite using $x = \frac{w(t)}{D}$

$$R(t) = xR_{\mathrm{on}} + (1 - x)\, R_{\mathrm{on}} \ , \tag{22}$$

we know that $0 \leq w \leq D \Rightarrow 0 \leq x \leq 1$, thus we see that $R(t) = R_{\mathrm{on}}$. But this involved setting R_{off} to R_{on} which contradicts assumption 3:

$$R_{\mathrm{on}} = R_{\mathrm{off}} \perp R_{\mathrm{on}} \ll R_{\mathrm{off}} \ . \tag{23}$$

□

2.3.3 Problems with the current

Theorem 2.3. *The substitution of V in equation 10 by IR is an incorrect expression for $\frac{dw(t)}{dt}$*

Proof. The zeroth assumption says that matter can not be created or destroyed in this system, thus we can derive Kirchhoff's laws. From Kirchhoff's laws, the total measured current (I) is a sum of all all the currents in the system, specifically the electronic current i_e and the ionic vacancy current i_v:

$$I(t) = i_v + i_e \ . \tag{24}$$

Thus the right hand side of equation 11 (now we are ignoring the issues with the resistance) should be

$$\mu_v \frac{R_{\mathrm{on}}}{D} \left(i_v + i_e\right) \ , \tag{25}$$

which is actually a measure of the average drift velocity of all the charge carriers in the system, $\mathbf{s}$, i.e. $\mu_v \frac{R_{\mathrm{on}}}{D} \left(i_v + i_e\right) = s$.

Thus

$$\mu_v \frac{R_{\mathrm{on}}}{D} \left(i_v + i_e\right) = \mathbf{s} \perp \mu_v \frac{V}{D} = \mathbf{s}_v, \tag{26}$$

unless $\mathbf{s} = \mathbf{s}_v$.

From equation 8, the average drift velocity of all charge carriers can be expressed as

$$\mathbf{s} = \left[\frac{n_v \mu_v}{N} + \frac{n_e \mu_e}{N}\right] L \ , \tag{27}$$

where n_v is the number of oxygen vacancy charge carriers in the system, n_e is the number of electron charge carriers in the system and N is the total number of charge carriers given by $N = n_v + n_e$.

$\mathbf{s} = \mathbf{s}_v$ iff

$$\frac{n_v \mu_v}{N} + \frac{n_e \mu_e}{N} = \mu_v \tag{28}$$

which can happen in two cases:

1. if $n_e = 0 \Rightarrow N = n_v \Rightarrow \frac{n_v \mu_v}{n_v} = \mu_v$
2. if $\mu_e = 0 \Rightarrow e$ are not charge carriers, $\Rightarrow N = n_v$.

Case 1 suggests that the memristor device is ionic rather than mixed ionic and electronic. This is not mathematically impossible, but does not fit with the physics of the system we are trying to model. Case 2 contradicts the known values [1] of the mobilities: $\mu_e = 0 \perp \mu_e > \mu_v$ and $\mu_e, \mu_v \not< 0$. $\square$

2.3.4 Problems with the Distance

It is possible that Strukov et al were attempting to describe only the part of the device that was doped (the 'on' part). In which case, they might have used V_{on} in equation 10 (it was not included in the derivation), where V_{on} refers to the potential dropped between $x = 0$ and $x = w$. This voltage is not the voltage measured in the circuit, nor is it possible to measure it in a real memristor in the set-up described in [19] (or in our experimental memristor measurement set-ups) however, it could conceptually exist in the equivalent variable resistor system shown in the 4th subfigure of figure 2a in [19].

Theorem 2.4. *Describing Only the 'On' Part of the Device Leads to Recursion*

Proof. If we are describing only the 'on' part of the device then equation 8 should have been

$$\frac{dw(t)}{dt} = \mu_v L_{\mathrm{on}} , \tag{29}$$

where L_{on} is the electric field over $0 \leq x \leq w$.

Thus, equation 10 should have been

$$\frac{dw(t)}{dt} = \mu_v \frac{V_{\mathrm{on}}(t)}{w(t)} , \tag{30}$$

where $w(t)$ is now the distance because this is the limit of the 'on' part of the device.

Taking the time integral of equation 30 gives w as a function of w i.e. $w = f(w)$ (the actual integral can not be given here as we do not have an espression for how w varies with time because this is what we were attempting to derive!) As in theorem 2.5, this is recursive.

$\square$

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[1] μ_e in this system is around $1 \times 10^{-6} cm^{-2} V^{-1} s^{-1}$ [25], μ_v is around $1 \times 10^{-10} cm^{-2} V^{-1} s^{-1}$ [19]

This approach is equivalent to taking the limit case of $R \sim R_{\text{on}}$. Assumption 3, $R_{\text{off}} \gg R_{\text{on}}$, implies that $R \sim R_{\text{off}}$ if we were to approximate it. And $R \sim R_{\text{on}} \perp R \sim R_{\text{off}}$ as in Theorem 2.2.

This concludes our disproof by exhaustion of the correctness of equation 11. To demonstrate that the following equations are unsupported, we need to prove that if we replace R_{on} with R we cannot derrive the expressions for memristance 16.

2.3.5 The Corrected Version of Equation 11 can not be Used to Derrive the Expression for Memristance in Equation ??

Theorem 2.5. *Using $V = IR$ to substitute for V in equation 10 leads to recursion.*

Proof. If we substitute R into equation 11 in place of R_{on}, we would get $w(t) = \mu_v \frac{R}{D} q(t)$ for equation 14, which implies that $w(R)$.

Equation 6 shows that $R(w)$.

Putting equation 13 into 6 is equivalent to $R(w(R))$. As $R(w)$ we could write $R(w(R(w)))$. As $w(R)$ we can write $R(w(R(w(R))))$. This can be expanded indefinitely, showing that the statement is recursive (and not a helpful step in attempting to calculate the memristance) and we can not solve this. ☐

Thus, we have shown that we can not proceed further with the derrivation and therefore that equation 16 can not been derrived from the steps in [19].

2.3.6 Consequences for the Strukov Derivation

There is a critical error in the derivation. The substitution of IR for V in deriving equation 11 is incorrect (theorem 2.1, both for the choice of the resistance (theorem 2.2) and current (theorem 2.3) and leads to incorrect device descriptions or contradictions (theorems 2.2 and 2.3). Doing the correct substitution for R leads to recursion (theorem 2.5) making it impossible to derive Strukov's expression of memristance and leaving this expression entirely unsupported. As a result, equations 11, 13 and 16 have to be thrown away.

These mathematical results are well supported by the chemistry of the system. The voltage is driving the motion of both charge carriers in the system and it is included in equation 10 performing this function, but applied to only the vacancies (as the voltage experienced by both charge carriers is the same). The voltage can also be calculated from total resistance and current, but the use of total current is not appropriate in an equation specifically and only about the vacancies. Perhaps the confusion arises from the use of the word ‘vacancy’ in the field. The vacancies that alter the resistance are oxygen ion vacancies (oxygen ions are drifting in the opposite direction to the vacancy current), they are not holes due to electron motion, and have a different drift velocity and dynamics to the electrons. Given that, it seems self-evident that an equation governing the dynamics of the vacancy drift should contain terms that all refer to the vacancies and not the total number of charge carriers. Integrating to get equation 13 means that the charge is the total charge, Q in the system. This charge is $Q = q_v + q_e$, which as $q_e \gg q_v$ means that the charge in equation 13 has been vastly overestimated. I suggest that the charge that should be in the memristor equation should be the charge relating to the vacancies and not the total charge within the system.

This misunderstanding of the value of q has led to a confusion about the value of the magnetic flux associated with memristance. If the relevant q is not $\int I dt$, then the magnetic flux is not $\int V dt$, and thus is unlikely to be as large, which explains why the magnetic flux has not been experimentally measured.

We will now go on to derive the memristance for the Strukov memristor.

3 Introducing the Memory Property

The following analysis is general but will be discussed using the example of the Strukov memristor. The question of whether the Strukov memristor is a true Chua memristor will be approached by first asking how a Chua memristor would behave based on Chua’s equations. We are going to focus on the physical property of the memristor responsible for its memory, which we shall call the ‘memory property’.

For there to be a memory in a memristor, we shall postulate that the memory property must be both separate from the conducting electrodes, and slower to respond to a voltage change than the conducting electrons. This slower response time leads to the lag in current which gives rise to the hysteresis loop and explains the frequency dependence of memristance: if the voltage changes too fast for the memory property, it can’t respond fast enough for a measurable change and the size of the hysteresis loop shrinks to a straight line (this is the ohmic regime).

The memory property has to respond to the voltage, which suggests that it either needs to be affected by the potential difference and therefore be charged, or to undergo a structural change due to the electrical energy supplied. Note that to make a memristor, rather than a memristive system, the device property that causes this change in resistance must be controlled by voltage – this is necessary for the memristor to be a two-terminal device which is part of Chua's definition for a fundamental circuit element memristor. And, for the memristor to be of any real use, this change in memory property has to be (at least qualitatively) reversible, so the device can switch back and forth.

There are several different possibilities for memory properties, such as charged ions in the PEO-PANI memristive system [15], or the concentration of spin electrons [26] in spintronic systems or the 'thermal' phase change that can be triggered by voltage in a VO_2 thin film [27].

To be explicit, the memory property is the physical property or part of the system which holds the system's state, and the state variable is the theoretical label of the aspect of the memory property which is q in equation 3. We expect these to be related.

Strukov et al assumed that q (the state variable in equation 3) should be the conducting electronic charge, q_e (or rather the total charge in the system which is over-whelmingly the electronic charge), as they derive it by integrating the ohmic electronic current over time, but this is impossible because the conducting electrons cannot travel more slowly than themselves and thus the electronic charge can not be a cause of hysteresis. The electronic current measures the lag, therefore it can not also cause it, otherwise there would be no lag to measure it would be indistinguishable from the response. Therefore, another aspect of the system must be responding to the voltage on a different timescale and it is this response which affects the electronic current.

The property that best fits the criteria outlined for the memory property is the oxygen vacancies because the state of the memristor is stored in the oxygen vacancies (as they do not dissipate when the voltage is removed), they drift slower than the conducting electrons which introduces the hysteretic lag (which is recorded in the electronic current because their presence changes the resistivity of the material) and they respond to voltage. If the vacancies are the memory property, then it is their charge, q_v, which is the state variable which should be used in Chua's equation. Thus, although it is the effect on the electronic current which is measured (and will be of use in real world devices), the electronic current is irrelevant in the actual process of memristance.

Note, Strukov et al, and others, were aware that the vacancies are the physical property responsible for the memory, however, they did not relate it to the state variable in Chua's equation, instead assuming that it was the electronic charge, the time integral of electronic current, which was important. Most improvements to the early phenomenological model [19] focused either on abstracting the behaviour, usually by entirely removing the contribution of the vacancies or by concentrating only on the behaviour of Chua's equations. These posit that approaches are incorrect for two reasons: A. the conducting electrons are not the charge that should be in Chua's equations; B. just focusing on Chua's equation does not give an adequate model for the behaviour of real devices.

To derive an alternative model of the memristance of the Strukov memristor, we shall calculate the magnetic flux associated with vacancy motion.

4 Calculating the Magnetic Flux

We are going to investigate Strukov et al's memristor as it is a typical memristor device and had the phenomenological theory applied to it. The schematic for this memristor is shown in figure 1. Note that the x direction is taken as being the direction of ion flow, with D being the limit of the titanium dioxide layer in the direction (i.e. it's thickness which is 10nm [19]) and $w(t)$ being the position of the boundary where $0 < w(t) < D$. The y and z axes are in the plane of the electrodes with the limits E and F and are both 50nm in the crossbar memristor. We take the terms of our integral in the coordinate system for inside the memristor, i.e.: r_x, r_y and r_z. To be explicit about our starting assumptions, we assume a linear boundary and that the memory property required from our analysis of Chua's equations is the oxygen ions/oxygen vacancies.

4.1 Calculating the Magnetic Feild Due to the Oxygen Vacancy Current

To calculate the flux which should be in Chua's equation, we start by calculating the flux associated with a steady-line current, and this is given by the Biot-Savert law for the magnetic field associated with a volume current. This is the most appropriate formulation of the Biot-Savert law because we are going to consider the magnetic flux just above the memristor surface where the memristor is best viewed as a 3-dimensional object. The Biot-Savert law comes from magnetostatics, a branch of theory that describes the magnetic effects due to constant currents, although our current will change, magnetostatics is still a valid approach because the changing current is, in this case, far slower than that to which such theory is successfully applied (namely mains A.C. (50-60Hz)) [28].

From this expression and using the Biot-Savert law, the magnetic field (also known as magnetic flux density), $\mathbf{B}$, at a point, p, associated with with this charge is given by the Biot-Savert law for a volume current, $\mathbf{J}$:

$$\mathbf{B}(p) = \frac{\mu_0}{4\pi} \int \frac{J\mathbf{d\hat{J}}\times\hat{\mathbf{r}}}{r^2}\mathrm{d}\tau \tag{31}$$

where μ_0 is the permittivity of a vacuum, $\mathrm{d}\hat{\mathbf{J}}$ and $\mathrm{d}\hat{\mathbf{r}}$ are the unit vectors for $\mathbf{J}$ and $\mathbf{r}$ where $\mathbf{r}$ is the vector of length r from the volume infinitesimal $\mathrm{d}\tau$ to point p, given by $\mathbf{r} = \{r_x\hat{\mathbf{i}}, r_y\hat{\mathbf{j}}, r_z\hat{\mathbf{k}}\}$.

The magnetic field integral in equation 32 is taken over the volume of the device that contains flowing vacancies (which is $w \times E \times F$). This volume is time-dependent due to w, but at an instant in time, t, the magnetic field is given by

$$\mathbf{B}(p,t) = \frac{\mu_0}{4\pi} \int_0^F \int_0^E \int_0^{w(t)} \frac{\mathbf{J}\times\mathbf{r}}{|\mathbf{r}|^3}\mathrm{d}r_x\mathrm{d}r_y\mathrm{d}r_z \;, \tag{32}$$

where we have expanded the volume integral to a 3-D Cartesian space and $|\mathbf{r}|^3$ is the cube of the length of vector $\mathbf{r}$, and the change in power of the denominator arises from the replacement of the unit vector in equation 31 with the definition of a unit vector (which is $\hat{\mathbf{r}} = \frac{\mathbf{r}}{|r|}$). Note that the integral in equation 32 is over the Cartesian components of $\mathbf{r}$, and to be explicit $|\mathbf{r}|^3 = (r_x^2 + r_y^2 + r_z^2)^{\frac{3}{2}}$

The integral is solved using the technique of integration by parts, taking the cross product in the numerator, $\mathbf{J}\times\mathbf{r}$, as $\mathrm{d}g(r_x, r_y, r_z)$ and the denominator, $\frac{1}{(r_x^2+r_y^2+r_z^2)^{\frac{3}{2}}}$ as $f(r_x, r_y, r_z)$ [2].

As we know the form of $\mathbf{r}$, to solve equation 32 we will need to know the volume current density vector, $\mathbf{J}$, it is given by $\mathbf{J} = \rho_v\mathbf{s}_v$, where ρ_v is the charge density of oxygen vacancies and $\mathbf{s}_v$ is their average drift velocity. The charge density can be expressed as $\rho_v = \frac{nz_v}{vol}$, where z_v is the charge on a single oxygen vacancy (+1 in this scheme because we are dealing with single oxygen vacancy in $\mathrm{TiO_2}$ material, not the equivalent motion of an oxygen ion $\mathrm{O^-}$ or the effective charge on an oxygen atom $\mathrm{O^{-2}}$), n_v is the number

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[2] The rule for integration by parts in this case is given by $\iiint f\left(r_x,r_y,r_z\right)\left(\frac{\partial}{\partial r_x}\frac{\partial}{\partial r_y}\frac{\partial}{\partial r_z}g\left(r_x,r_y,r_z\right)\right)\mathrm{d}r_x\mathrm{d}r_y\mathrm{d}rz =$
$f\left(r_x,r_y,r_z\right)g\left(r_x,r_y,r_z\right)$
$-\iiint\left(\frac{\partial}{\partial r_x}\frac{\partial}{\partial r_y}\frac{\partial}{\partial r_z}f\left(r_x,r_y,r_z\right)\right)g\left(r_x,r_y,r_z\right)\mathrm{d}r_x\mathrm{d}r_y\mathrm{d}rz$

of oxygen vacancies and *vol* is the volume. We substitute this for ρ_v and substitute for s_v using equations 8 and 9, as in section 2.2 Thus, the volume current density for all the oxygen vacancies is

$$\mathbf{J} = \frac{n_v z_v \mu_v \mathbf{L}}{vol} \ . \tag{33}$$

Note that L and μ_v are average properties, so we are dealing with the bulk movement of vacancies: individual vacancies can move at different speeds and in different directions, but drift along the feild lines on average. The total charge due to the oxygen vacancies (and also our memory property), q_v, is $q_v = n_v z_v$ and so our final equation for $\mathbf{J}$ is

$$\mathbf{J} = \frac{q_v \mu_v \mathbf{L}}{vol} \ , \tag{34}$$

and is function of time because $q_v(t)$. Note that $\mathbf{L}$ can also vary with time in some experiments. The vector $\mathbf{J}$ is taken as being

$$\mathbf{J} = \{\frac{q_v \mu_v L}{vol}\hat{\mathbf{i}}, 0\hat{\mathbf{j}}, 0\hat{\mathbf{k}}\} \ , \tag{35}$$

because field and drift direction are taken as being in the $+x$ direction for the Strukov device.

If we put equation 34 into equation 32 and solve as described above we get

$$\mathbf{B}(p) = \frac{\mu_0}{4\pi} L \mu_v q_v \{P_x, -xzP_y, xyP_z\} \tag{36}$$

with

$$\begin{aligned} P_x &= 0 \, , \\ P_y &= \frac{F}{2\left(w^2+E^2+F^2\right)^{\frac{3}{2}}} \\ &\quad -\frac{1}{2wEF}\frac{a_y}{\left(\left(w^2+F^2\right)b\right)} \\ &\quad +F\arctan\left(\frac{wE}{F\sqrt{w^2+E^2+F^2}}\right) , \end{aligned}$$

and

$$
\begin{aligned}
P_z \quad = \quad & \frac{E}{2\left(w^2+E^2+F^2\right)^{\frac{3}{2}}} \\
& -\frac{1}{2wEF}\frac{a_z}{\left(\left(w^2+E^2\right)b\right)} \\
& +E\arctan\left(\frac{wF}{E\sqrt{w^2+E^2+F^2}}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
a_y \quad = \quad & wE(F^2\left(E^2+F^2\right)^2+w^4\left(2E^2+F^2\right) \\
& +w^2\left(2E^4+5E^2F^2+2F^4\right)) \ , \\
a_z \quad = \quad & wF(E^2\left(E^2+F^2\right)^2+w^4\left(E^2+2F^2\right) \\
& +w^2\left(2E^4+5E^2F^2+2F^4\right)) \\
b \quad = \quad & \left(E^2+F^2\right)\left(w^2+E^2+\ F^2\right)^{\frac{3}{2}} \ .
\end{aligned}
$$

P_y and P_z contain only the dimensions of the memristor, so even if they are not analytically simple, they are easy to calculate numerically. As expected of a magnetic field, the divergence of the field is zero, i.e. $\nabla\cdot\mathbf{B}=0$.

As an example, for a Strukov memristor which is close to being full with the maximum number of vacancies (i.e. the limit) the field at point p is given by

$$\mathbf{B}(p)=\{0,-6.37q_vVxz,6.37q_vVxy\},$$

where V is the applied voltage, $p=\{x,y,z\}$ and x, y and z refer to a second set of coordinates which are located outside the memristor whose unit vectors are $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ [3]. The curl of $\mathbf{B}$ is non-zero as the field curls around the current in an anti-clockwise direction. An example curl for the

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[3] The volume current is constrained within the memristor and can be written in terms of coordinates inside the memristor. The magnetic field (as caused by the volume current) can only exist outside the memristor and therefore can be written in terms of coordinates from outside the memristor. The two sets are labelled differently here to avoid confusion. If the distinction is not made between the two sets, then it's possible that the inside coordinates might be integrated over twice, which would be wrong. Perhaps confusingly, the limits are the similar. The inside coordinates have the limits: $0\leq r_x\leq D$; $0\leq r_y\leq E$; $0\leq r_z\leq F$;. The outside coordinates can go from $-\infty$ to ∞ but must avoid the volume within the memristor

system above evaluated at $\{0, 0, 0\}$ (just inside the left hand side of the device) is $\nabla \times \mathbf{B} = \{12.74 q_v V, -6.37 q_v V, -6.37 q_v V\}$. The gradient of the field indicates the direction of travel that gives maximal field values, i.e. $\nabla \mathbf{B} = \{0, -6.37 q_v V x z, 6.37 q_v V x y\}$, namely that there is no increase in the x direction and that the maximal vector field is experience by looping around the x axis.

4.2 Calculating the Magnetic Flux due to Oxygen Vacancy Current

The magnetic $\mathbf{B}$ field is the magnetic flux density and so to calculate the magnetic flux through a surface associated with this field, φ, we need to take the surface integral

$$\varphi = \int \mathbf{B} \cdot \mathrm{d}\mathbf{A} \tag{37}$$

where $\mathrm{d}\mathbf{A}$ is the normal vector from the surface infinitesimal $\mathrm{d}A$.

As it is a surface integral, to calculate the magnetic flux we need to pick a surface to evaluate over. It makes sense to choose a surface that correlates to one of the surfaces of the device. Picking the surface just above the device ($0 < x < D$, $0 < y < E$, $z = F$ [4]), we use the surface normal area infinitesimal, $\mathrm{d}\mathbf{A}$, which is given by $\mathrm{d}\mathbf{A} = \{0, 0, \hat{\imath}\hat{\jmath}\}$. As is standard in electromagnetism, we integrate over the entire area. The limits of the surface are taken to be the dimensions of the device.

By putting the expression for $\mathbf{B}$ in equation 36 into equation 37 and taking the surface integral, we derive the general form of the magnetic flux passing through a surface i-j:

$$\varphi = \frac{\mu_0}{4\pi} L \mu_v i j P_k q_v \ , \tag{38}$$

where $i \epsilon \{x, y, z\}$, $j \epsilon \{x, y, z\}$, $k \epsilon \{x, y, z\}$, i.e. P_k is component in the vector in equation 36 which is perpendicular to the surface $i - j$ where i and j can be any two of the Cartesian directions.

Equation 38 contains a physical magnetic flux, satisfies Chua's equation $\varphi = M(q)q$ [9] and crucially has been derived without reference to Chua's equations. Note that this relation between charge and flux in a memristor includes the material properties and is the first to do so.

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[4] Actually $z = F + dF$ so the surface is just above the memristor, avoiding any surface effects.

By reference to equation 4, the Chua memristance in this device is expressed as:

$$M\left(q_v\left(t\right)\right) = UX\mu_v P_k\left(q_v\left(t\right)\right)\ , \tag{39}$$

where we have gathered up the constants and explicitly included P_k's dependence on q_v.

Equation 39 can be considered as three separate parts:

1. U, the universal constants: $\frac{\mu_0}{4\pi}$, this term includes the effects of the permittivity of a vacuum on memristance. It's inclusion in the equation clearly demonstrates that magnetism is involved in memristance.

2. X, the experimental constants: DEL, where DE is the surface the flux was calculated over as we've substituted in for i and j, in this case the top of the device. The constant X will be different for different devices and experiments and is time-dependent if V is.

3. β, the material variable: $\mu_v P_z$, this includes the physical dimensions of the device, but it will change throughout the experiment as a result of the moving boundary, $w(t)$, whose motion is caused by the drift of vacancies across the device. This is the only term that contains variables. Note, it is from this term, via the value of μ_v and its interaction with the applied voltage frequency that the memristor's frequency dependence arises.

For the Strukov memristor, P_y and P_z are equal in magnitude because the magnetic field is centro-symmetric around the vacancy current (which flows in the $+x$ direction, see figure 1). Thus, the values of the memristance calculated from the x-y and x-z surfaces are the same, see Table 1. As w is a measure of how far the vacancies extend into the material it is dependent on q_v and thus P_k is a function of q_v. Interestingly, equation 39 implies that the Chua memristance has directional dependence, and will only be non-zero for surfaces that cut magnetic field lines, the y-z surface doesn't, and thus P_x is zero. This raises the intriguing possibility of memristance being best described as a three-dimensional property. For most systems there will be only one non-zero value. As Chua suggested that the memristance could be either charge or flux controlled [2], the memristance calculated here should be capable of being controlled by either and thus holding a memory of either. And it does, $P_k(q_v)$ is part of the Chua memristance which holds the memory of the charge, β, is part of the memristance which holds the memory for the flux.

Device surface	Area infinitesimal $\mathrm{d}\hat{\mathbf{A}}$	Integral	Value for Strukov memristor
Top	$\{0, 0, \hat{\imath}\hat{\jmath}\}$	$\varphi_{\text{top}} = \int_0^E \int_0^D \mathbf{B} \cdot \mathrm{d}\hat{\mathbf{A}} \mathrm{d}x \mathrm{d}y$	-3.186$\times10^{-15} q_v$
Bottom	$\{0, 0, -\hat{\imath}\hat{\jmath}\}$	$\varphi_{\text{bottom}} = \int_0^E \int_0^D \mathbf{B} \cdot \mathrm{d}\hat{\mathbf{A}} \mathrm{d}x \mathrm{d}y$	-3.186$\times10^{-15} q_v$
Front	$\{0, \hat{\imath}\hat{k}, 0\}$	$\varphi_{\text{front}} = \int_0^F \int_0^D \mathbf{B} \cdot \mathrm{d}\hat{\mathbf{A}} \mathrm{d}x \mathrm{d}z$	-3.186$\times10^{-15} q_v$
Back	$\{0, -\hat{\imath}\hat{k}, 0\}$	$\varphi_{\text{back}} = \int_0^F \int_0^D \mathbf{B} \cdot \mathrm{d}\hat{\mathbf{A}} \mathrm{d}x \mathrm{d}z$	3.186$\times10^{-15} q_v$
Left	$\{\hat{\jmath}\hat{k}, 0, 0\}$	$\varphi_{\text{left}} = \int_0^F \int_0^E \mathbf{B} \cdot \mathrm{d}\hat{\mathbf{A}} \mathrm{d}y \mathrm{d}z$	0
Right	$\{-\hat{\jmath}\hat{k}, 0, 0\}$	$\varphi_{\text{right}} = \int_0^F \int_0^E \mathbf{B} \cdot \mathrm{d}\hat{\mathbf{A}} \mathrm{d}y \mathrm{d}z$	0

Table 1: Table for the magnetic flux as calculated from the different possible surfaces of the memristor.

Putting in real-world values for the device characteristics (as above, including $V = 1V$) for the Strukov memristor gives a memristance equation of $d\varphi = 3.53 \times 10^{15} dq$, and a $\varphi - q$ plot is linear over the range $0 < w < D$ (where w must be strictly more than 0 to avoid 1/0 errors), indicating this model is a perfect memristor because it satisfies Chua's constitutive definition (equation 3) with a constant value.

With these real-world example values, the Stukov memristors has flux of 2.44$\times10^{-29}$Wb. In contrast, the magnetic flux associated with the conducting electrons through the same surface[5] is -4.07$\times10^{-24}$Wb. This is in the opposite direction and approximately 100 000 times bigger than the va-

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[5] To get the number of electrons, we've assumed that the TiO_2 is acting like a metal and every titanium atom is giving up a conducting electron. To get the number of oxygens that can be lost, we're assuming that maximum of 3% of available oxygen atoms

cancies' magnetic flux. This may explain why the magnetic flux associated with memristor function has not been experimentally measured.

4.3 Strukov 'Magnetic Flux' Term

The units of P_y and P_z are m^{-2}, and interestingly, Strukov et al's model included an approximation for the material parameter β as $\beta_{st} \approx \mu_v/D^2$. β_{st} can be viewed as an unintentional approximation for the magnetic flux as both β terms have units of Wb^{-1}. However, this approximation is not quantitative: in order to produce the flux expected from Strukov et al's model, the memristor would need to contain a **B** field similar in size to those found in neutron stars and magnetars. Also, Strukov et al's model also lacks any reference to magnetic permittivity (of vacuum or the material), the inclusion of which would be expected in a system describing magnetic flux.

Proof. If $\varphi = \frac{D^2}{\mu_v}$ and $\varphi = \int B.dA$, and D^2 taken to be an approximation for the surface of the memristor xy, ie the area we integrate over, then $B = \frac{d\varphi}{dA}$. As $dA = dxdy$,

$$B = \frac{d^2\varphi}{dxdy} = \frac{d^2\left(\frac{D^2}{\mu_v}\right)}{dxdy} = \frac{d^2\left(\frac{xy}{\mu_v}\right)}{dxdy} = \frac{1}{\mu_v}\ , \tag{40}$$

which is approximately $1{\times}10^{14}$ T, magnetars (neutron stars with strong magnetic fields) have magnetic fields from 10^8T upwards. □

5 Memory and Conservation Functions

How can the tiny ($\sim 10^{-29}$Wb) magnetic flux in the Strukov memristor be associated with the large effect seen in experimental I-V curves? The answer is because the memristive movement of charge affects the resistivity of the material, and it is this resistivity change that is 'sampled' by the conducting electronic current.

When measuring a memristor it is conventional to measure the electronic current, not the ionic current. As the electronic current is many times larger and faster than the movement of vacancies, we can even choose to ignore the vacancy contribution to the total flow of charge, without introducing a significant error. What is needed is the memristance as experienced by the conducting electrons, $R_{\mathrm{tot}}(t)$. The component of that memristance which is directly due to the changing resistivity of the doped material, M_e, which we shall call the Memory Function as it encapsulates the memristor's memory, is given by

$$M_e = CM(q_v(t)) \ ,$$

where C is an experimentally determined parameter for the material.

Because the ion mobilities of the electrons, μ_e, and the vacancies, μ_v, are measured experimentally, it is predicted that $C = (q_e\mu_e) \, / \, (q_v\mu_v)$.

The memory function describes the doped part of the titanium dioxide, $TiO_{(2-x)}$, as experienced by the electrons traversing it. The proportion of the memristor made up of this form changes, and, because matter must be conserved in the model, the proportion of the memristor made up of un-doped TiO_2 is given by the conservation function, R_{con}, which is simply the resistance of the un-doped material:

$$R_{\mathrm{con}}\,(t) = \frac{(D - w\,(t))\,\rho_{TiO_2}}{EF} \tag{41}$$

where ρ_{TiO_2} is the resistivity of un-doped TiO_2. Note, Strukov et al's model was based on a similar conservation function (as it arises from Ohm's law) and, as this is responsible for most of the observed change in the device, their model gave memristor $I - V$ curves.

The total resistance as experienced by the conducting electrons, R_{tot}, is then given by

$$R_{\mathrm{tot}} = R_{\mathrm{con}} + M_e \ . \tag{42}$$

As R_{tot} is a resistance that changes with time due to the action of charge it is therefore also a memristance and this equation gives the pinched hysteresis loop in I-V space which is indicative of memristance, as shown in figure 3. Separately, both the conservation and memory functions are also memristances and both can give rise to a memristive I-V curve. The memory function is just the Chua memristance expressed in terms of the conducting electrons. The conservation function is memristance due to the change in volume of the un-doped material, which is the second effect of the oxygen vacancies' movement into the TiO_2.

One definition of a Chua memristor is that it is a function of a single state variable [2]. The only variable in the conservation function is w and because w is a measure of how far the memristive charges have moved, the Chua memristance, and thus the memory function, can be written in terms of w instead of q. Therefore, R_{tot} can be written as a function of w only, thus demonstrating that the Strukov memristor is a Chua memristor with one state variable w. Assuming that the vacancies are spread out in the

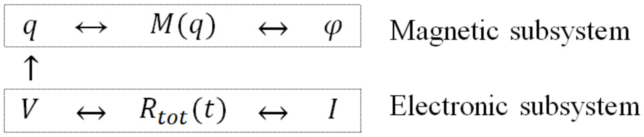

Figure 2: Diagrammatic representation of the presented memristor model. The Magnetic subsystem refers to the magnetic flux required by Chuas definition of memristance and the non-electronic charge carriers that give rise to it. The Electronic subsystem refers to the effects experienced by the conducting electrons. Shown here is an example of a voltage controlled memristor where the applied voltage causes the non-electronic carriers to drift and their movement effects local resistivity of the material, altering the total resistance R_{tot} and affecting the measured electronic current.

same way across the device, (ie that $\mathrm{TiO}_{(2-x)}$) w is a measure of q and these equations can also be expressed in terms of a single state variable q.

Thus we have demonstrated that in order to describe memristance, two systems need to be considered, as is shown diagrammatically in figure 2. The first is the 'electronic' system, which is associated with the conducting electrons and which provides the 'electronic current' response to an applied voltage.

The second system is the 'magnetic' system, which contains the magnetic flux and the 'memristive' charge, i.e. the vacancies. Note that these charge carriers are not especially magnetic (neither is it claimed here that the memristive charge carriers are acting as magnetic monopoles, although that comparison has been made [29]). Instead, the charge responsible for the memory function of the memristor is being separated conceptually from the charge due to the conducting electrons. It is important to realise that the existence of memristive magnetic flux in the system does not mean that the memristor is magnetised in a traditional sense. The 'magnetism' in the system is not similar to the magnetism of ferrous materials that are capable of holding or reacting strongly to a magnetic field. Instead the memristor magnetic effect is similar to the atomic scale magnetic susceptibility as understood and exploited by NMR spectroscopy and MRI imaging. Furthermore, the 'magnetic' system does not describe all of the properties of the memristor that exhibit magnetism. For example, there is magnetic flux associated with the conducting electrons, but this flux is mostly irrelevant to understanding the memristive operation of the device.

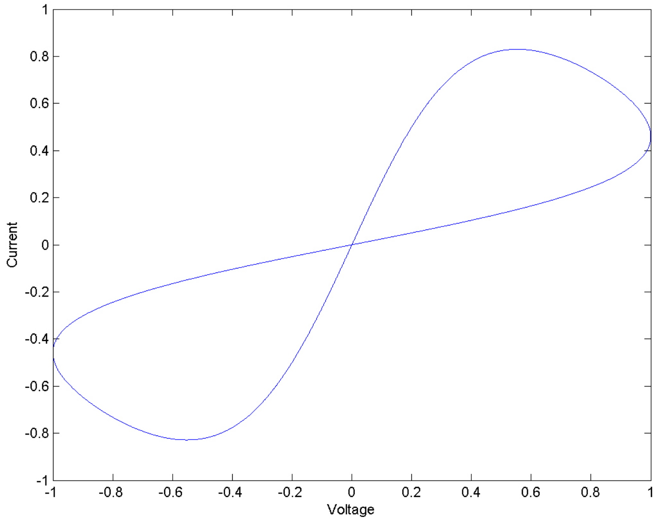

Figure 3: An example memristor I-V curve as calculated from the presented theory. The memory allows there to be more than one possible current for a given voltage, which causes the hysteresis, and as memristors are passive devices and therefore can not store energy, the current must be zero when the voltage is, which causes the distinctive pinched shape. This I-V curve matches Chuas theoretical I-V curves – real memristors tend to have a threshold voltage below which the system less memristive, which causes elongated, pinched areas in the I-V curve.

6 The effect of dimensionality

The numerical values for the magnetic flux as predicted from this model and Strukov's model are vastly different. In terms of the variables, the model presented here requires a consideration of both the vacancy charge and the electronic current. Theoretically, there is also a difference in dimension. Strukov et al's model is 1-dimensional as it is only dependent on the spatial dimension D, ie only the thickness of the device matters. This dependence on D has led to the claim that memristors will only work on the nanoscale [19], a claim that has been contradicted by the creation of macroscopic memristors such as Gale et al's thick film memristors [23]. The theory presented here in this paper it three-dimensional and thus the effect of electrode width ought to change the Chua memristance and the properties of the device (Experimental tests of this are given in [30]).

6.1 Modelling flexible sol-gel memristors

To compare a 3-dimensional description with a 1-dimensional one, we shall briefly consider sol-gel devices fabricated with 40nm thick titanium dioxide and 4mm wide aluminium electrodes (see [31]). As there is no change in orientation, the numbers can simply be put into the equations above. A sol-gel memristor of this type has a magnetic $\mathbf{B}$ field that is 2.3×10^{10} Teslas smaller than the Strukov memristor. The increased thickness of the titanium dioxide layer reduces the field by only an order of magnitude, the rest of the reduction is due to the extra width of the electrodes, something which is not included in one-dimensional models (so applying the Strukov model to this device would give answers that are inaccurate).

The size of the electrode surface also affects the calculation of the flux, as we calculate it over the whole surface, which gives a value of $\varphi = 1.76777 * 10^{-19}q$ which is four orders of magnitude smaller than the values given in table 1. If both devices were charging up to the maximum charge (i.e. approaching the limit as described above) the sol-gel device has 1.62×10^{7} more Webers of flux than the Strukov memristor. This is because the device has a larger volume (note that the electrons' magnetic flux is also increased by a similar amount). If we had only considered the thickness of the layer, we would get the erroneous result that the sol-gel memristor had a smaller amount of flux than the Strukov memristor. This clearly demonstrates that, although the 1-dimensional model is a useful simplification, it does ignore necessary aspects of the system.

6.2 Modelling PEO-PANI memristors

Further differences between theoretical approaches are observed when devices with different geometry are considered. The PEO-PANI memristor (also called the organic memristor) is difficult to model with a one-dimensional electron-based theory. The electrons flow along the PANI layer, parallel to the glass substrate and the resistance of the PANI is affected by the lithium ions which are drifting between the PEO and PANI perpendicular to the glass substrate, see figure 4 and [14, 15, 32, 33] for further details. This is obviously a system that can not be accurately described in one-dimensional space. The second complication is that the ions that cause the resistivity change are obviously chemically-unrelated to the electrons so describing this device without reference to these ions would involve tortuous logic.

As above we shall take the x axis as being the direction that the electrons drift in and thus the lithium ions drift parallel to the z axis, so w is now

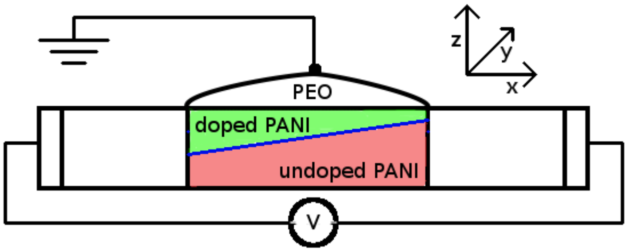

Figure 4: Device schematic for the plastic (PEO-PANI) memristor. A layer of poly-aniline (PANI) is applied before lithium (Li^+-doped Poly-ethylene oxide (PEO)) is placed on top and earthed. Ionic dopants diffuse from the PEO into the PANI making it conductive, application of voltage through the PANI layer has the effect of moving the dopants in and out of the PANI layer.

related to the z axis, i.e. $0 < w < F$. The width of the active area is now 1mm ([34]), which is also the extent of the device in the y direction. Thus, J, the ionic current density is given by

$$\mathbf{J} = \frac{q_v \mu_{Li} \mathbf{L}}{vol} = \{0, 0, \frac{q_v \mu_{Li}}{E^2 w} \hat{k}\} \, .$$

where the active area is taken to be a square of E^2. The resulting magnetic field is zero in the z-direction and has expressions in the x and y directions analogous to those shown above and there is no magnetic flux through the top and bottom of the device.

The conservation function has to be written from the point of view of the electrons, so the formulae are 90 degrees out compared to the Strukov memristor and we must also include the unswitchable resistance, R_u, of the part of the memristor that is un-switchable (shown in white in figure 4) because it does not have PEO on top of it, this is given by.

$$R_{\mathrm{u}} = \frac{(D - E)\rho_{\mathrm{on}}}{EF} \, .$$

The switchable volume in the off state, R_{off}, of the memristor is given by $\frac{E\rho_{\mathrm{off}}}{Ew}$, which cancels to $\frac{\rho_{\mathrm{off}}}{w}$ and thus the conservation function is $R_{\mathrm{u}} + R_{\mathrm{off}}$ and the memory function is as above.

As the electronic and ionic currents are at 90° to each other, their magnetic fields can interact. Because the electrons have the larger magnetic field which has a negative z-term, a positive ions in the device should drift downwards quicker than it drifts upwards, as the electrons' magnetic field works with the applied electrical field. This would lead to fast kinetics for recharging the device (the PEO-PANI memristor starts off doped and is switched to undoped and then re-doped during an I-V cycle), which is indeed what is reported in [5].

7 Conclusion

We have demonstrated that the Strukov derivation has a mathematical error that renders the expression for $M(q)$ in [19] unsupported and unusable. It has been shown that calculating φ from q_v gives an alternative expression for $M(q)$ that fits the real system and satisfies Chua's constitutive definitions in [9]. This approach clarifies the value of memristor magnetic flux and demonstrates that the Strukov memristor is a Chua memristor. By doing so, it relates the previously theoretical q and φ to real-world material properties for the first time. This result will provide a new paradigm in memristor research, by conceptually separating the memristive magnetic flux from the conducting electronic current, and by introducing the concept of a memory property. Further work in this direction involves relating the fabrication and experimental values found in the theory to actual device behaviour. For our work in this direction see [30].

The model presented here is spatially three-dimensional and it has been demonstrated that this makes a significant difference in the theoretical predictions for real world systems such as the flexible TiO_2 sol-gel and PEO-PANI memristors. Investigations into whether a three-dimensional theory is borne out by experiment is included in [30].

This model is for 'perfect' Chua memristors. Further work would be to apply this theory to other memristor systems including those with different memory properties (i.e. not ions), memristive systems (i.e. those with two state variables) and ReRAM. Preliminary work applying this theory to filamentary memristors has been published in [35].

The theory presented in this paper gives rise to some intriguing possibilities. The 'conversion' between the Chua memristance described from the point of view of the oxygen vacancies and the memory function as described from the point of view of the electrons suggests that resistance is a quantity that depends on the charge carrier experiencing it. Reformulating resistance in this way could provide more natural ways to discuss systems with many charge carriers, such as the PEO-PANI memristor and living systems. Many memristors have the presence or absence of ions as a memory property, and with this work the expectation for a large magnetic flux to be associated with memristance has been removed, which suggests that synapses are actually biological memristors rather than just being conveniently modeled by the mathematics.

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