Secular Increase of Atronomical Unit from Analysis of the Major Planet Motions, and its Interpretation

G.A. Krasinsky and V.A. Brumberg
Institute of Applied Astronomy, Russian Academy of Science, Kutuzov Quay 10, 191187, St. Petersburg, Russia, email: kra@quasar.ipa.nw.ru

Celestial Mechanics and Dynamical Astronomy **90:** *267–288, 2004.*

Abstract. From analysis of all available radiometric measurements of distances between the Earth and the major planets (including observations of martian landers and orbiters over 1971-2003 with the errors of few meters) the positive secular trend in the Astronomical Unit AU is estimated as $\frac{d}{dt}AU = 15 \pm 4$ m/cy. The given uncertainty is the 10 times enlarged formal error of the least–squares estimate and so accounts for possible systematic errors of measurements and deficiencies of the mathematical model. The reliability of this estimate as well as its physical meaning are discussed. *A priori* most plausible attribution of this effect to the cosmological expansion of the Universe turns out inadequate. A model of the observables developed in the frame of the relativistic background metric of the uniform isotropic Universe shows that the corresponding dynamical perturbations in the major planet motions are completely canceled out by the Einstein effect of dependence of the rate of the observer's clock (that keeps the proper time) on the gravitational field, though separately values of these two effects are quite large and attainable with the accuracy achieved. Another tentative source of the secular rate of AU is the loss of the solar mass due to the solar wind and electromagnetic radiation but it amounts in $\frac{d}{dt}AU$ only to 0.3 m/cy. Excluding other explanations that seem exotic (such as secular decrease of the gravitational constant) at present there is no satisfactory explanation of the detected secular increase of AU, at least in the frame of the considered uniform models of the Universe.
Keywords: cosmology–ephemerides–relativity

1 Introduction

At present, positional observations of the major planets make it possible to control the scale factor of the solar system (Astronomical Unit AU) with the accuracy about one meter and even better. The progress has started at the end of seventies when the first precise measurements (with the errors of few meters) of distances between the Earth and landers placed on Mars by the space missions Viking and Pathfinder have been obtained. The ongoing program of exploration of Mars by martian orbiters provides even more accurate observations. It seems natural to expect that a secular trend of AU if discovered might reveal some fundamental features of the space-time. Indeed, the well established cosmological expansion of the Universe manifests itself on the large space scales as a Doppler effect in form of the Hubble redshift in spectra of galaxies or quasars. It poses a question whether this expansion on the space scale of the solar system generates the analogous Doppler shift (involving the corresponding detectable secular increase of distances) in the radiometric observations of the major planets. Intuitively one could expect that the value of AU should increase in a secular way with the rate of the order of the Hubble constant. Considering the Hubble age of the Universe of about 13 billion years one might expect that $\frac{d}{dt}AU \approx 1$ km/cy. However, this value is too large and definitely is ruled out by positional observations of the major planets. In somewhat more sophisticated approach the cosmological effects in the heliocentric motion of the planets are calculated by using the standard metric of the four–dimensional time-space of the uniform Universe (Masreliez 1999). The resulting effects seem to be rather paradoxical. In terms of the length unit that rises with time in the expanding Universe, the derived AU should decrease while the mean motions of the planets increase (such an effect of the apparent Solar System shrinking is called the cosmic drag). The predicted accelerations in the planetary longitudes are also too large varying from $1.4''/cy^2$ for the Earth to $5.8''/cy^2$ for Mercury. However, constructing the internationally adopted DE ephemerides, no such large perturbations in the motion of the major planets have been detected. Noteworthy that the dynamical model developed by Masreliez (1999) is incomplete as it ignores an important feature of the rigorous equations of motion of the test particle orbiting the Sun in the expanding Universe, namely, the apparent secular decrease of the gravitational constant G.

Beneath we present more adequate model obtained as a solution of the corresponding Einstein's field equations. This approximate solution generalizes the well–known Schwarzschild solution to the case of the attracting singularity (the Sun) in the background field of the expanding Universe. This field may correspond to any uniform distribution of matter (e.g., to the Friedmann metric of the open Universe or the flat metric of the expanding Universe). Rigorous solution of this type has been constructed by McVittie (1933) revealing the mentioned above effect of the cosmic drag in the so called cosmical coordinate system (the distant quasars and galaxies are in rest in respect to this system). According to McVittie, such effect vanishes in observer's (locally statical) coordinates. However more detailed investigation shows that even in such a system the cosmic drag still exists. It is important to note that the cosmic drag is a coordinate dependent effect and a correct prediction of observable consequences of the cosmological expansion on the planet motion should take into account not only the dynamical aspects of the cosmic drag but also the corresponding

Einstein effect in the light propagation. In the case of the expanding Universe this effect means a secular trend between the observer's proper time and the coordinate time (the latter being the argument of the relativistic equations of motion). Our considerations show that the large contributions to observables from these two sources completely cancel out each other. Hence, the expansion of the Universe in principle cannot be detected analyzing any observations of the solar system bodies, at least in the frame of the models of the uniform Universe. One may find in the literature the statements (without taking into account the Einstein effect) that the expansion of the Universe does not affect the planetary motion (see, for instance, Järnefelt (1940, 1942)). This statement has not ever been rigorously proved. Moreover, it is wrong if the Einstein effect is ignored.

There exists another source of perturbations in the solar system that really leads to the secular increase of the Astronomical Unit at a marginally detectable level. This effect is caused by the loss of the solar mass M due to the light emission and solar wind. The rate $\dot{M}$ of the loss caused by the solar wind is estimated as $\dot{M}/M \approx 3 \cdot 10^{-12}/cy$ (Suniaev 1986) giving for the resulting secular increase of AU the value about 0.3 meters per century which is at the threshold of sensitivity of the contemporary radiometric observations of martian space probes. The loss of the solar mass for radiation is several times less and may be disregarded.

At last, we still cannot rule out the possibility that the gravitational constant decreases secularily in time. It also might be detected from analysis of the radiometric observations of the space probes.

The paper consists of the two parts. In the theoretical section we show that the expansion of the uniform Universe does not produce any measurable effects in the motion of the major planets. This result is not self–evident because the cancellation of the large dynamical effects with those of the light propagation needs rather delicate analytical considerations to be proved. In the second part we discuss some results of estimating $\frac{d}{dt}AU$ from analysis of the radiometric observations. The experimentally derived value of $\frac{d}{dt}AU$ indeed appears positive but its value is at least by one order larger than that predicted for the assumed rate of the loss of the solar mass.

This work seems to be a first attempt to detect such fine effects from analysis of the most accurate radiometric observations available at present, and our results may be considered as preliminary ones. Because the cosmological expansion of the uniform Universe is proved to do not affect the value $\frac{d}{dt}AU$, further experimental studies of this parameter from analysis of the accumulating dataset of the radiometric observations seem to be very promising in several aspects including detection of secular decrease of the gravitational constant (if the solar physics sets limits on the rate of the loss of the solar mass). From the other hand, theoretical studies of dynamical effects in the Solar System caused by non-uniform features of the expanding Universe also seem to be perspective.

2 Cosmological expansion and planetary motion in the expanding Universe

2.1 Time–space metrics and equations of planetary motion

If the singularity approximating the solar gravitation is absent then the Friedmann metric of the open expanding Universe with the uniform mass distribution can be presented in the conformally Galilean form as follows (Fock 1955; Landau and Lifshitz 1967):

$$ds^2 = A\left(c^2 dt^2 - d\mathbf{r}^2\right), \tag{1}$$

where

$$A = \left(1 - \frac{q}{S}\right)^4, \qquad S = \sqrt{c^2t^2 - r^2}, \tag{2}$$

c is the light velocity, $r^2 = x^2 + y^2 + z^2$, $d\mathbf{r}^2 = dx^2 + dy^2 + dz^2$, and q is a constant parameter of the metric.

We have chosen the conformally Galilean coordinates because to study the planetary motions they are more suitable than the 'cosmical' co–moving coordinates commonly used in cosmology.

For this study it is convenient to assume that the origin of the space coordinates coincides with the center of the Sun. The second parameter of the metric is the value of S_0 of S for the present epoch at the solar center. Hence, the zero point of the time scale is the epoch of the singularity when $S = ct = 0$ at the solar center. The parameter $S_0 = ct_0$ is proportional to the age of the Universe t_0.

It is known (Fock 1955; Landau and Lifshitz 1967) that the Hubble constant H_0, measured at the present epoch, may be expressed in terms of q , S_0 as follows:

$$H_0 = \frac{c}{S_0 A_0^{3/4}}\left(1 + \frac{q}{S_0}\right), \qquad A_0 = \left(1 - \frac{q}{S_0}\right)^4.$$

The second relation presents the mean density ρ_0 of the matter in the Universe (at present) in terms of the same parameters q , S_0

$$\rho_0 = \frac{3q}{2\pi G S_0^3 A_0^{3/2}}, \tag{3}$$

where G is the gravitation constant.

For applications to planetary dynamics, the metric (1), (2) has to be generalized to include perturbations caused by the solar potential. It is known that the metrics of the gravitating point of the mass M with sufficient accuracy (not too close to the singularity) may be presented as follows:

$$ds^2 = \left(1 - \frac{2m}{r}\right) c^2 dt^2 - \left(1 + \frac{2m}{r}\right) d\mathbf{r}^2 , \qquad m = \frac{GM}{c^2} .$$

To take into account the gravitational singularity (the Sun), the perturbed Friedmann metric should satisfy the Einstein field equations with the right part being the sum of the dust matter tensor and that of the singularity at the origin of the coordinates. Both gravitational fields are weak and in spite of the nonlinear structure of the field equations these equations may be linearized resulting to superposition of the both fields with some small but important corrections. It is shown in Appendix that the metric obtained as a solution of the perturbed Einstein equations may be written in the form

$$ds^2 = \left(A - \frac{2m}{r}\sqrt{A}\right) c^2 dt^2 - \left(A + \frac{2m}{r}\sqrt{A}\right) d\mathbf{r}^2. \tag{4}$$

Deriving the quasi–Newtonian equations of motion in the field (4), the factor at $d\mathbf{r}^2$ may be set equal to A. If the trajectory of the test body is parameterized by $\mathbf{r}(\tau)$, $t(\tau)$ in terms of the proper time τ then the equations of motion may be written in Lagrangian form (making use of the vectorial symbolic)

$$\begin{aligned} \frac{d}{d\tau}\left(\frac{\partial L}{\partial \mathbf{r}_\tau}\right) - \frac{\partial L}{\partial \mathbf{r}} &= 0, \\ \frac{d}{d\tau}\left(\frac{\partial L}{\partial t_\tau}\right) - \frac{\partial L}{\partial t} &= 0, \end{aligned}$$

where Lagrangian L is given by the expression

$$L = \left(A - \frac{2m}{r}\sqrt{A}\right) t_\tau^2 - A\frac{|\mathbf{r}_\tau^2|}{c^2}$$

and the lower index τ denotes differentiating with respect to τ.

This system has the integral $L = C$. Without loss of generality the constant C may be set equal to 1. Then this integral can be used to transform the system to the coordinate time t as the independent variable. In this way the equations for the space coordinates $\mathbf{r}$ are separated from the equation relating the coordinate time with the proper time. Omitting simple transformations, we present the final differential equations in the form (consistent, also, with Eqs. (4.3.38) of Brumberg (1991))

$$\ddot{\mathbf{r}} = -\frac{GM}{\sqrt{A}\, r^3}\mathbf{r} + \frac{2q}{S^3 A^{1/4}}(|\dot{\mathbf{r}}|^2 - c^2)(\dot{\mathbf{r}}t - \mathbf{r}).. \tag{5}$$

For the purely uniform Universe (i.e. if $M = 0$) the equations of motion have three-parametric family of the partial solutions

$$\mathbf{r} = \dot{\mathbf{r}}t, \tag{6}$$

meaning that the test particle at the position $\mathbf{r}$ moves in the radial direction with the Hubble velocity $\mathbf{v} = \mathbf{r}/t$ where t is the time interval elapsed from the epoch of singularity. This family as a whole presents rectilinear motions of the uniformly distributed field of particles (galaxies) generating the metric of the expanding Universe.

In application to the solar system bodies the equations (5) may be significantly simplified. The time variable t explicitly entering into the right member of these equations is reckoned from the epoch of the singularity. Hence, its value at the present epoch is equal to the Universe age: $t_0 \approx 17.7 \times 10^9$ years. We set $S = ct$ neglecting the light propagation time in the solar system (about five minutes for AU and several hours for maximal light crossing) in comparison with the Universe age. We may also neglect the ratio $|\dot{\mathbf{r}}|^2/c^2$ of the order 10^{-8} for any planet of the solar system. As a result, the equations of motion of the test particle in the field of solar gravity at the background of the expanding uniform Universe reduce to the form

$$\ddot{\mathbf{r}} = -\frac{Gm}{A^{1/2} r^3}\mathbf{r} - \frac{2q}{ct^3 A^{1/4}}(\dot{\mathbf{r}}t - \mathbf{r}). \tag{7}$$

With sufficient accuracy we may set

$$A = \left(1 - \frac{q}{ct}\right)^4,$$

and restrict ourselves by the planar case with $z = \dot{z} = 0$.

Hence, we ignore the dependence of S on the space coordinates $\mathbf{r}$ in the expression (2) assuming $S = S_0 + c(t - t_0)$ at least for the time span of several hundred years. Indeed, the error of this approximation is proportional to the square of the ratio of the light interval r/c (about five minutes) to the Universe age t_0. Therefore, with very high accuracy the parameter A of the metric may be presented in the form

$$A = A_0 + \dot{A}_0(t - t_0)$$

where

$$A_0 = \left(1 - \frac{q}{S_0}\right)^4, \qquad \dot{A}_0 = 4\left(1 - \frac{q}{S_0}\right)^3 \frac{cq}{S_0^2}. \tag{8}$$

These expressions should be substituted into the first term of the right part of equations of motion (7) while for the small second term the approximation $A = A_0$ is quite sufficient. As the result the equations of motion of the test particle around the Sun at the background of the uniform Universe reduce to the simple form as follows:

$$\ddot{\mathbf{r}} = -\frac{GM}{\sqrt{A_0} r^3}\left(1 - 2\frac{t - t_0}{\mathrm{T}}\right)\mathbf{r} - \frac{2}{\mathrm{T}}\dot{\mathbf{r}} \tag{9}$$

where

$$\mathrm{T} = \frac{S_0}{c}\left(\frac{S_0}{q} - 1\right) \tag{10}$$

is the time–like parameter that characterizes the perturbing effects in the planetary motion due to the cosmological expansion.

Note that the *a priori* value of the ratio $q/S_0 < 0.1$ is small enough and thus relation (3) shows that the parameter S_0/c is close to the Hubble age of the Universe $1/H_0$. Then from definition (10) of T one can see that T exceeds the Hubble age at least by one order. Thus the expected perturbations are less than one could anticipate from the naive assumption that the Hubble age is the characteristic time of the evolution of the planetary orbits due to the cosmological expansion. This is true for the considered open model with $\rho_0 < \rho_{cr}$ where ρ_{cr} is the critical mass–density

$$\rho_{cr} = \frac{3H_0^2}{8\pi G}.$$

It may be shown, however, that for the flat expanding model with $\rho_0 = \rho_{cr}$ (consistent probably with recent observations, see (Spergel et al. 2003)) the Hubble time is indeed the characteristic evolutional time for the corresponding equations of motions. But as shown below it does not involve any observational consequences.

Dependence of the first term in the right member of equations (9) on the constant parameter A_0 does not lead to any observable effects and may be disregarded by putting $M^* = M/\sqrt{A_0}$ resulting in the equations

$$\ddot{\mathbf{r}} = -\frac{GM^*}{r^3}\left(1 - 2\frac{t - t_0}{\mathrm{T}}\right)\mathbf{r} - \frac{2}{\mathrm{T}}\dot{\mathbf{r}}\,. \tag{11}$$

Solution of differential equations (11) written for the heliocentric motion of the Earth has to be inserted to the right member of expression (4) enabling one to obtain the relationship between the coordinate time t and the proper time τ realized on the Earth as the atomic time scale

$$\left(\frac{d\tau}{dt}\right)^2 = A - \frac{2m}{r}\sqrt{A} - A\frac{v^2}{c^2}.$$

If the cosmological expansion is not considered ($A = 1$) then this relation presents relativistic effect in the atomic time and is used in processing observations of the planets in a standard way. Thus we may disregard the terms proportional to m and v^2/c^2. The first term results in the following relation between the coordinate time t and the atomic time τ of an observer (setting the constant factor $\sqrt{A_0}$ equal to 1 by rescaling the time unit of the proper time τ):

$$\frac{d\tau}{dt} = \sqrt{A} \approx 1 + 2\frac{t - t_0}{\mathrm{T}}. \tag{12}$$

2.2 Dynamical effects in the expanding Universe

From differential equation (11) one can see that the expanding Universe produces the effect similar to the secular degrease of the gravitational constant G with the rate $\dot{G}/G$ given by the expression

$$\frac{\dot{G}}{G} = -\frac{2}{\mathrm{T}},$$

in which the time-like constant T is defined by relation (10).

Introducing the polar coordinates in the plane of motion $z = 0$

$$x = r\cos\lambda\,, \qquad y = r\sin\lambda \tag{13}$$

the equations (11) can be transformed into

$$\ddot{r} - r\dot{\lambda}^2 = -GM^*\left(1 - 2\frac{t-t_0}{T}\right)\frac{1}{r^2} - \frac{2}{T}\dot{r}\,, \qquad \frac{1}{r}\frac{d}{dt}(r^2\dot{\lambda}) = -\frac{2}{T}r\,\dot{\lambda}\,. \tag{14}$$

The second equations admits the straightforward integration

$$r^2\dot{\lambda} = na^2\exp\left(-2\frac{t-t_0}{T}\right) \tag{15}$$

or by neglecting the higher orders with respect to the small parameter $(t-t_0)/\,\mathrm{T}$

$$\dot{\lambda} = n\left(\frac{a}{r}\right)^2\left(1 - 2\frac{t-t_0}{T}\right), \tag{16}$$

a being an arbitrary constant related with the longitude frequency n by the third Kepler's law $n^2a^3 = GM^*$. Then the first equation of (11) takes the form

$$\ddot{r} = \frac{n^2a^4}{r^3}\left(1 - 4\frac{t-t_0}{T}\right) - \frac{n^2a^3}{r^2}\left(1 - 2\frac{t-t_0}{T}\right) - \frac{2}{T}\dot{r}\,. \tag{17}$$

Hence, the motion of the test body can be represented by the approximate expressions for the radius–vector and longitude

$$r = a\left(1 - 2\frac{t-t_0}{T}\right), \tag{18}$$

$$\lambda = \lambda_0 + n(t-t_0)\left(1 + \frac{t-t_0}{\mathrm{T}}\right). \tag{19}$$

These relations should be used for perturbations of both the observed planet and the Earth. If they are not taken into account then the experimentally derived value of the Astronomical Unit AU would reveal the secular trend

$$\frac{d\,AU}{AU} = -2\frac{t-t_0}{\mathrm{T}}\,. \tag{20}$$

Hence, if positional observations of the planets are processed without considering in the dynamical model the perturbations of the radius–vector and longitude given above then it results to the negative secular trend (20) in the observed values of AU and the positive quadratic term (19) in longitudes of all planets (the cosmic drag effects). In fact the effect in the longitudes means that the time scale of the observer needs the quadratic in time correction.

However, these dynamical effects do not cover all consequences of the cosmological expansion for the major planet motion. Not accounted effects of the same character arise if in modeling the observables one ignores a small but important cosmological effect in the light propagation from the terrestrial observer to the planet. This effect is caused by the rate clock difference between the observer's atomic time scale and the time scale of the equations of motion (Einstein effect).

2.3 Einstein effect for the light propagation in the expanding Universe

To model the observed distances from the observer to the planet the following reductions for the light propagation should be carried out.

1. If t_1, t_2, t_3 are the instants of the signal emitting, its reflecting from the target, and its receiving back at the terrestrial station, respectively, then the radar time delay dt is equal to $t_3 - t_1$ and the modeled values of dt, t_1, t_2, t_3 are obtained by iterations from the relations

$$\begin{aligned} dt \equiv t_3 - t_1 &= c^{-1}(|\mathbf{r}_p(t_2) - \mathbf{r}_e(t_1)| + |\mathbf{r}_e(t_3) - \mathbf{r}_p(t_2)|), \\ t_2 &= t_1 + c^{-1}|\mathbf{r}_p(t_2) - \mathbf{r}_e(t_1)|, \\ t_3 &= t_2 + c^{-1}|\mathbf{r}_e(t_3) - \mathbf{r}_p(t_2)|, \end{aligned}$$

where $\mathbf{r}_p(t), \mathbf{r}_e(t)$ are heliocentric positions of the target and the observer at the corresponding moment t of the coordinate time scale of planetary ephemerides.

Several iterations are needed to calculate the moments t_1, t_2, t_3 and the time delay dt. Afterwards, to match the modeled time delay dt with its measured value dt_{obs}, the value dt should be transformed to the scale of the proper time τ multiplying it by the scale factor $1 + d\tau/dt$ with the value (12) for the derivative $d\tau/dt$. Thus the calculated time delay dt is related with the observed quantity dt_{obs} as follows:

$$dt_{obs} = dt\left(1 + 2\frac{t - t_0}{\mathrm{T}}\right).$$

If this reduction is not applied then in addition to the dynamical trend (20) the value of AU derived from observations would contain the positive secular trend

$$\frac{d}{dt}AU = 2\frac{t - t_0}{\mathrm{T}}. \tag{21}$$

It is seen that contributions (20) and (21) cancel out each other.

2. Before entering into the ephemerides given in the coordinate time scale, the proper time τ reckoned by the observer's clock has to be transformed into the coordinate time t. This transformation is obtained by integrating the equation (12). If this reduction is not carried out then the observed mean longitude would contain the additional negative term quadratic in time and proportional to the mean motion n

$$d\lambda = -\frac{n(t-t_0)^2}{\mathrm{T}}. \tag{22}$$

The correction (22) has to be added to the right member of relation (19). It is seen that the dynamical and kinematical effects in the mean longitude also cancel out as well in AU.

Thus, we have shown that the effects of expanding uniform Universe do not involve any measurable effects in the motion of the major planets. This result is proved for the open model of the uniform Universe when $\rho < \rho_{cr}$ (or $q < S_0$) but it holds true for the case of the flat expanding Universe. It may be proved by extending the above analysis to the case of the metric with $\rho = \rho_{cr}$. For brevity such consideration is not presented here.

3 Secular increase of the Astronomical Unit: observed evidences

3.1 Observations

For the aim of this work only radiometric observations of the major planets have been used, as the most accurate ones. These observations make it possible to derive positions of the planets without relating them to the stellar or quasar background (that could be produced only by adding angular observations of more poor accuracy). To study the effect of the secular variation of the Astronomical Unit this approach seems to be quite adequate. We use the database of the radiometric observations of the major planets of the Institute of Applied Astronomy compiled and supported by Pitjeva (2001, 2003). It includes all radiometric data processed in construction of the ephemerides DE405 which are the internationally adopted standards. The data are published by Standish, the author of the DE ephemerides, at the web site *http : /ssd.jpl.nasa.gov/iau − comm4/* of the Jet Propulsion Laboratory. This dataset has been appended by Pitjeva with radar observations of Mercury, Venus and Mars made at the radar station Eupatoria (Crimea) since 1961, and at American radar stations in 1961–1966 (ranging to Mercury and Venus), and in 1967–1971 (ranging to Mars) which are not included into the JPL database due to their lower accuracy in comparison with the more recent data. The measurements of ranging to surfaces of the planets are reduced for effects of topography making use of either the available hypsometric data (for Mars and Venus), or so called closing points (for Mercury). Observations distributed in one–day intervals are combined to normal points with the resulting errors varying from several kilometers for the old data to about 150 meters for the more recent ones.

Since 1971 a great improvement in accuracy of the measured distances to Mars has been achieved as the result of exploration of Mars by space missions. These data are of the two types: firstly, the measured distances to the landers on the surface of Mars (Viking 1 in 1976–1992, Viking 2 in 1976–1977 and Pathfinder in 1997), and secondly, the distances to the martian barycenter synthesized from results of tracking of space probes orbiting around Mars (Mariner–9, 1971–1972, Mars Global Surveyor (MGS), 1998–2003, and Odyssey, 2002–2003). *A priori* errors of the lander data are about 5 m. For the spacecraft data they vary from 10–15 m for Mariner–9 to 2–3 m for MGS and Odyssey.

There is also a large volume of phase measurements of distances to Mars (about 15000 normal points for Vikings 1, 2 and 7500 normal points for Pathfinder mission). They are of a centimeter level of accuracy which could not be realized in the full scale due to impossibility to resolve the phase ambiguity and to tailor the phase measurements made at different days. That is why the observations of this type were used only in form of differences of the phases for two moments of time separated by two-minute time interval, transformed to the length unit. In this way the phase ambiguity vanishes, but the obtained differences lose sensitivity to the long–periodic effects caused by errors in the orbital elements. On the other hand, these differential measurements are very sensitive to the short periodic effects of the Mars rotation and have been used here mainly to strengthen the solution with respect to the coordinates of martian landers and parameters of rotation of Mars.

The total dataset of the measurements consists of about 7500 measured ranging distances (individual or reduced to normal places) to Mercury, Venus and Mars (either to sub–radar points on the surfaces, or to the landers, or else to the barycenter of Mars), and about 23000 of the Doppler-like auxiliary measurements. All observations were weighted in accordance with the *a priori* accuracy of the individual measurements or normal places assigned by Pitjeva. The great volume of the Doppler–like differential measurements (given by the phase differences) was downweighted to avoid influence of their systematic errors.

The observation span of 1961-2003 has allowed to estimate the centennial secular trend of AU with the formal error about 0.4 meter.

3.2 Dynamical model and ephemerides

For more reliability of results the described above observations have been processed with two independent dynamical models that could provide the adequate level of accuracy. The first model is presented by the ephemerides DE405 adopted as an international standard (Standish 1998). These ephemerides are distributed in the form of Chebyshev polynomials and hence they cannot be applied to calculate partial derivatives with respect to a number of parameters that have to be estimated simultaneously with the secular effects under consideration. The partials have been calculated with the numerical theory EPM, developed in the Institute of Applied Astronomy of St-Petersburg, making it possible to improve all the parameters with the both ephemerides and then to compare the results obtained. Numerical integration of the EPM ephemerides is performed within the framework of the relativistic (post–Newtonian) time–space metric of General Relativity following the basic ideas of the DE ephemerides as they are outlined in (Newhall et al. 1983). Since then significant developments have been made to refine the models of the orbital motion of the major planets, as well as of the orbital and rotational motions of the Moon for the aims of successful processing more recent and more precise observations of distances to the major planets and the Moon (Krasinsky, 1999, Pitjeva 2001, Krasinsky, 2002).

The most serious problem encountered in constructing up-to-date ephemerides of the major planets is the strong necessity to take into account the perturbations from a large number of minor planets with poorly known masses. In the EPM ephemerides the integrated differential equations include (in addition to the equations for the major planets from Mercury to Pluto and for the Moon and its librations) also the equations of motion of 297 minor planets. For the biggest five of them mutual perturbations are accounted while for the others only the perturbations from the major planets. Masses of the minor planets for the integration have been obtained by the method described in (Krasinsky et al. 2001, 2002). This method follows basically that used in the DE ephemerides and may be briefly outlined in the following way. The minor planets are separated to three sets in accordance with their taxonomic classes: C (Carbonic), S (Sillicum) and M (Metallic) derived from the taxonomic codes taken from NASA database SBN ('Small bodies node of the NASA Planetary Data System', http://pdssbn.astro.umd.edu). This database contains also the radii of the asteroids resulted from IRAS (Infra Red Astronomical Satellite) mission. Assuming appropriate densities for each of the taxonomic classes, one obtains values of the masses of all asteroids involved in numerical integration. Mean densities of each of the taxonomic classes are considered as parameters to be improved from general fitting of the radiometric measurements, simultaneously with other parameters which include also masses of the biggest five asteroids (estimating them from perturbations in the orbits of the major planets). To take into account the overall effect from the large number of remaining small asteroids whose individual perturbations could not be calculated by the direct numerical integration, these perturbations are approximated by the model of a solid ring in the ecliptic plane. The mass of this ring also are included into the set of the estimated parameters. For more details of evaluating the asteroid masses see (Krasinky et al, 2001; Krasinsky et al, 2002).

At the level of accuracy of the best radiometric observations available now, uncertainties of the adopted masses of the major planets also influence the modeled observables. To rule out this source of systematic errors in the secular trend of AU, the masses of Venus, Mars, Jupiter and Saturn are included into the list of parameters under estimation.

Dynamical effects of the uncertainty of the lunar mass are negligible but the geometrical displacement of the Earth relative to the barycenter of the Earth–Moon system (so called lunar term) is very sensitive to the lunar mass. Therefore, the ratio of the masses of the Moon and Earth is considered as one more unknown of the analysis. Any error in the mass of the Moon gives rise to a monthly term in the residuals and so does not affect the secular rates under study but its error might increase the noise of the residuals.

Both the DE405 and EPM ephemerides account for the dynamical effects of the coefficient J_2 of the second harmonics of the solar potential. The value $J_2 = 2.0 \times 10^{-7}$ adopted in DE405 is used. On the whole our analysis of the radiometric observations has confirmed this value, and moreover, it appears that J_2 may be determined even more exactly if included into the list of the solve-for parameters.

The applied luni–planetary integrator is based on the Everhard method of numerical integration with the fixed step of 0.5 day. It is built into the programming system ERA developed to support scientific research of various kind in the field of Dynamical and Ephemeris Astronomy. The system is described briefly in (Krasinsky and Vasilyev 1997), the full Manual is available via anonymous FTP: *quasar.ipa.nw.ru/incoming/era/era7.ps*. Both the numerical integration and comparison of the constructed dynamical theory EPM with the observed radiometric distances to the planets have been carried in the frame of this system. The process may be independently reproduced with corresponding programs available in the mentioned above FTP server.

3.3 Estimation of the parameters

The total list of the estimated parameters is as follows:

- three Keplerian elements of the Earth (semi-major axis, excentricity and perihelion longitude) considering fixed the other three elements which relate the coordinate system of the major planets with the celestial reference frame,
- 6×4=24 orbital elements of Mercury, Venus and Mars,
- 3×3 =9 areocentric coordinates of the martian landers (Viking 1, Viking 2, and Pathfinder),
- 3 Euler's angles of the equator of Mars relative to its orbit, and 3 secular rates of these angles (6 unknowns),
- masses of the biggest five asteroids (5 unknowns),
- masses of Venus, Earth, Mars, Jupiter, Saturn, and of the ring of small asteroids (6 unknowns),

- ratio of the masses of the Moon and the Earth,
- mean radii of Mercury, Venus, and Mars (for Venus and Mars they are rather scale factors of the hypsometric maps used for reduction of ranging to these planets, 3 unknowns),
- parameters of the solar corona model for martian orbiters (one or two parameters for each upper conjunction for various solutions),
- densities of the asteroids of classes C and S (2 unknowns),
- coefficient J_2 of the solar potential,
- Astronomical Unit AU,
- secular rate of AU,
- quadratic in time term of the observer's time scale.

The basic solution includes these 62 parameters plus a bias of the normal points of Pathfinder, as well as the bias for Odyssey data (considered to be less reliable than the MGS ones). The solution has provided a satisfactory fit to the observations, as one can see from Table 1 in which the weighted random mean square errors (WRMS) of the post–fit residuals in the one–way distances are given for each group of observations, both for DE405 and EPM ephemerides.

Table 1. Statistics of the post-fit residuals

Observations	Dates	N	WRMS (m) DE405	WRMS (m) EPM
MGS	1998	315	3.1	4.1
MGS	1999	1786	1.1	1.1
MGS	2000	188	1.6	1.4
MGS	2001	1024	1.0	0.9
MGS	2002	717	1.4	1.3
MGS	2003	905	1.6	1.1
Odyssey	2002	775	1.3	1.3
Odyssey	2003	940	1.1	1.0
Viking 1	1976–1982	1172	9.6	8.6
Viking 2	1976–1977	80	5.7	5.3
Pathfinder	1997	90	2.6	2.4
Mariner 9	1971–1973	644	37.8	37.6
Mars ranging	1965–1995	403	725.0	733.0
Venus ranging	1961–1995	1355	637.0	647.0
Mercury ranging	1969-1997	706	1329.0	1328.0
Viking phase	1976-1078	15030	0.035	0.035
Pathfinder phase	1997	7613	0.003	0.003

One can see that the WRMS for the two dynamical models are practically identical. The derived estimates of the secular trend in AU are presented in Table 2 for a number of solutions in which $\frac{d}{dt}AU$ is estimated from various subsets of the observations (the statistics of the residuals presented in Table 1 corresponds to the basic solution). To make judgement on reliability of these results it is important to note that the correlations of $\frac{d}{dt}AU$ with other parameters of the basic solution do not exceed 0.5. On the other hand, the quadratic term in the planetary longitudes strongly correlates with the orbital elements of Mars, the correlations amounting up to 0.85. Thus, we believe that at present the estimate of $\frac{d}{dt}AU$ is much more reliable than that of the quadratic term in the planet longitudes.

Our experience shows that the estimate of $\frac{d}{dt}AU$ is sensitive to values of parameters of the solar corona model for the martian orbiters. Unfortunately, unlike the two–frequency observations of the landers, those of the orbiters are carried out only at one frequency. The published observations of the orbiters have been already reduced for the corona effects making use of some model, parameters of which are determined from a global processing, but with the secular trend of AU not estimated. In this way the reduced observations of the orbiters might acquire some systematic errors to absorb a part of effects of this trend. We believe that the theory-dependence of the published observations of the orbiters could be weakened if the corrections to parameters of the solar corona are included to the list of unknowns. If that is not done, the trend $\frac{d}{dt}AU$ appears smaller (but still positive); however such estimate seems not be very reliable, due to the mentioned methodological deficiencies of this approach. That is why in the basic solution the corrections to the model of the solar corona are considered as solve-for parameters, notwithstanding that the observartions of the orbiters have been already reduced for the solar corona.

Table 2 demonstrates that all the solutions tried give significant positive values of the secular rate $\frac{d}{dt}AU$, even excluding the data from orbiters. The estimates of the rate are given only for the ephemerides EPM, those obtained with DE405 are very close to them. For instances, the basic solution gives for EPM the secular trend in AU equal to 16.7 ± 0.41, while with DE405 the estimate is 16.5 ± 0.41.

Taking into account uncertainties of the results due to different systematic errors we believe that the most realistic estimate of the rate is

$$\frac{d}{dt}AU = 15 \pm 4 \text{ m/cy}. \tag{23}$$

Table 2. Estimates of secular rate of AU

$\frac{d}{dt}AU$ (m/cy)		Solution	Time span
16.7	± 0.4	All observations[(1)]	1961-2003
9.7	± 0.2	All observations[(2)]	1961-2003
7.9	± 0.2	All observations[(3)]	1961-2003
9.7	± 0.3	Landers + orbiters[(3)]	1971-2003
14.4	± 1.2	Ranging + orbiters[(3)]	1961-2003
61.0	± 6.0	Ranging + landers	1961-1997
21.3	± 5.6	Landers	1971-1997

(1) Estimating parameters of corona for orbiters, and masses of planets.
(2) The same, excluding parameters of corona for orbiters.
(3) Excluding masses of the major planets and the parameters of corona.

4 Discussion of the results

Theoretical interpretation of the derived positive non–zero estimate of the rate $\frac{d}{dt}AU$ meets serious problems. We may consider the following hypotheses.

1. The effect is an artifact caused by the systematic errors in observations or by omitting some effects in the dynamical model or in propagation of the radiosignal. Indeed as noted above this estimate of the rate is affected by difficulties of adequate modeling of the dispersive features of the solar corona when observing the martian orbiters near upper conjunctions. However, any reasonable values of the solar corona model do not eliminate the significant positive secular rate of AU. Solutions in which the adopted masses of the major planets from Venus to Jupiter are simultaneously improved show that $\frac{d}{dt}AU$ is only slightly affected by uncertainties of these masses. The estimate of $\frac{d}{dt}AU$ appears rather robust if one tries various sets of the estimated parameters. Therefore, we believe that the observed large positive value of $\frac{d}{dt}AU$ is not affected by deficiency of either the dynamical model or that of propagation of radio signal because probably all known sources of errors are accounted. On the other hand, one could expect that the non–zero $\frac{d}{dt}AU$ should be accompanied by the non–zero secular decrease of the planetary mean motion n of the same order in the parameter $dl = AU\,\dot{n}/n$. If such effect indeed does not take place, it could be supposed that the trend in AU is caused by some instrumental biases of the measured distances. However the parameter dl turned out strongly correlated with a number of parameters under estimation, especially with the areocentric longitudes of the martian landers, and cannot be considered as reliably determined. In any case it always was too small to match the derived value of $\frac{d}{dt}AU$. The constraint $dl < 5$ m/cy seems to be the most reliable estimate, and at present, we still cannot rule out the hypothesis that the derived rate of AU is an artifact due to instrumental biases in the observed data.

2. The main theoretical result of this paper is that the secular rate of AU cannot be explained by the cosmological expansion of the Universe in the frame of any model with uniform mass distribution. The zero value of $\frac{d}{dt}AU$ for such model is a fine balance between the dynamical effect of the cosmic drag and the Einstein effect in the light propagation. Since each of these effects is very large (in particular, for the model with $\rho = \rho_{cr}$ considered now as the most plausible) one might conjecture that taking into account the mass anisotropy of the Universe would lead to non–complete cancellation of these large effects resulting in some non–zero value of $\frac{d}{dt}AU$. At present it is not clear how to come to this difficult theoretical problem.

3. Permanent loss of the solar mass due to the electromagnetic radiation and solar wind involves a secular increase of AU. The largest contribution to the loss of the solar mass is due to the solar wind. To calculate the corresponding secular increase of AU we can apply the method of Section 2 used for equation (9) to the equation

$$\ddot{\mathbf{r}} = -GM\left[1 - \frac{\dot{M}}{M}(t - t_0)\right]\frac{\mathbf{r}}{r^3}$$

and after analogous transformations (with the assumed value of the loss due to the solar wind $\dot{M}/M \approx 3 \cdot 10^{-12}/cy$) we obtain

$$\frac{d}{dt}AU = -AU\,\frac{\dot{M}}{M} = 0.3 \text{ m/cy}$$

which is less than the observed value (23) by almost the two orders.

Probably the large value (23) derived from the observations cannot be reconciled with any physically meaningful model of the loss of the solar mass. Note that even now the formal error of the secular trend in AU only insignificantly exceeds the predicted rate due to the loss of the solar mass and in near future this effect must be accounted while constructing the ephemerides of the planets.

4. Other conjectures. Decrease $\dot{G}$ of the gravitational constant G with the rate $\dot{G}/G \approx -2 \times 10^{-12}$/year might explain the observed secular rate of AU. This cojecture seems to be the lest plausible.

In conclusion, we would like to underline that the by–product of precise measurements of the distances between the Earth and Mars provided by the ongoing program of exploration of Mars seems to be very promising for obtaining more reliable estimates of $\frac{d}{dt}AU$. Even now the formal error of this value is comparable with the expected effect from the solar wind. Accumulation of the data of this tipe makes it possible to derive more reliable value of $\frac{d}{dt}AU$. Then any deviations of the observed $\frac{d}{dt}AU$ from the predicted value of the rate of loss of the solar mass will be very informative either for the solar physics or for understanding deep features of the space–time evolution of the Universe.

References

Brumberg, V.A. 1991: Essential Relativistic Celestial Mechanics (Bristol: Hilger)

Fock, V.A. 1955: Theory of Space, Time and Gravitation (Moscow, in Russian)

Infeld, L., and Plebanski, J. 1960: Motion and Relativity (New York: Pergamon Press)

Järnefelt, G. 1940: Ann. Acad. Sci. Fennicae, A 45, 3

Järnefelt, G. 1942: Ann. Acad. Sci. Fennicae, A 45, 12

Krasinsky, G.A. 1999: Celes.Mech.&Dyn.Astron., 75, 39

Krasinsky, G.A., and Vasilyev, M.V. 1997: In Dynamics and Astrometry of Natural and Artificial Celestial Bodies, eds. I.M.Wytrzyszczak, J.H.Lieske & R.A.Feldman, (Kluwer), 239

Krasinsky, G.A., Pitjeva, E.V., Vasilyev, M.V., and Yagudina, E.I. 2001: Communications of IAA RAS, 139, (Saint–Petersburg: IAA)

Krasinsky, G.A., Pitjeva, E.V., Vasilyev, M.V., and Yagudina, E.I. 2002: Icarus, 158, 98

Krasinsky, G.A. 2002: Communications of IAA RAS, 148, (Saint–Petersburg: IAA) Landau, L.D., and Lifshitz, E.M. 1967: Theoretical Physics, Vol. 2, Field Theory (Moscow, in Russian)

Lightman, A.P., Press, W.H., Price, R.H., and Teukolsky, S.A. 1975: Problem Book in Relativity and Gravitation (New Jersey: Princeton)

Masreliez, C.J. 1999: Ap&SS, 266, 399

McVittie, G.C. 1933: MNRAS, 93, 325

Newhall XX, Standish E.M., and Williams J.G. 1983: A&A, 150

Pitjeva, E.V. 2001: Celes.Mech.&Dyn.Astron., 80, 249

Pitjeva, E.V. 2003: Communications of IAA RAS, 157, 3 (Saint–Petersburg: IAA)

Spergel D.N., Verde L., et al. 2003: ApJ, in press

Standish, E.M. 1998: JPL Interoffice Memorandum, 312.F-98-048, JPL

Suniaev, R. A. (editor) 1986: Physics of Cosmos. Sovetskaya Enziklopedia (Moscow, in Russian)

Appendix. Attracting mass at the cosmological background

We derive below the metric (4) describing a weak gravitational field of a single attracting mass (the Sun) at the cosmological background of the uniform isotropic Universe. The standard (Robertson–Walker) form of this background in the spherical co—moving coordinates reads (Landau and Lifshitz 1967)

$$ds^2 = c^2 d\xi^2 - a^2[d\chi^2 + \Sigma^2(d\theta^2 + \sin^2\theta d\varphi^2)] \qquad (A.1)$$

with

$$\Sigma = \begin{cases} \sin\chi\,, & \text{closed model} \\ \chi\,, & \text{flat model} \\ \sinh\chi\,, & \text{open model} \end{cases} \qquad (A.2)$$

and scalar factor a determined by the Einstein field equations. Considering that

$$d\chi^2 = \frac{d\Sigma^2}{1-k\Sigma^2}\,, \qquad k = \begin{cases} +1\,, & \text{closed model} \\ 0\,, & \text{flat model} \\ -1\,, & \text{open model} \end{cases} \tag{A.3}$$

and replacing

$$\Sigma = \frac{\rho}{1+\frac{1}{4}k\rho^2} \tag{A.4}$$

one often makes use of (A.1) in form of

$$ds^2 = c^2 d\xi^2 - \frac{a^2}{\left(1+\frac{1}{4}k\rho^2\right)^2}\left[d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)\right] \tag{A.5}$$

but this form is not convenient for our purposes. Another modification of (A.1) more convenient in our case is produced by the transformation

$$cd\xi = ad\eta\,. \tag{A.6}$$

There results:

$$ds^2 = a^2(\eta)[d\eta^2 - d\chi^2 - \Sigma^2(d\theta^2 + \sin^2\theta d\varphi^2)]. \tag{A.7}$$

As well known, for the models with zero pressure the world time ξ, radius a and the Hubble constant $H = a^{-1}da/d\xi$ have values (Landau and Lifshitz 1967)

$$\xi = \begin{cases} \frac{2q}{c}(\eta - \sin\eta), \\ \frac{q}{3c}\eta^3\,, \\ \frac{2q}{c}(\sinh\eta - \eta), \end{cases} \qquad a = \begin{cases} 2q(1-\cos\eta), \\ q\eta^2\,, \\ 2q(\cosh\eta - 1), \end{cases} \qquad H = \begin{cases} \frac{c}{2q}\frac{\sin\eta}{(1-\cos\eta)^2}\,, \\ \frac{2c}{q}\eta^{-3}\,, \\ \frac{c}{2q}\frac{\sinh\eta}{(\cosh\eta-1)^2} \end{cases} \tag{A.8}$$

for $k = 1$ (closed model), 0 (flat model) and -1 (open model), respectively, q being a scale factor.

The general Robertson–Walker metric (A.7) can be reduced to the conformally Galilean metric (e.g., problem 19.8 of Lightman et al. 1975). In case of the open model one can apply a simpler reduction resulting to the conformally Galilean metric in the form used by Fock (1955). Introducing

$$r = d\sinh\chi\,, \qquad ct = d\cosh\chi\,, \qquad q\exp\eta = d\,, \qquad \tanh\chi = \frac{r}{ct} \tag{A.9}$$

one transforms (A.7) (with $k = -1$) to the form

$$ds^2 = A[c^2dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)] \tag{A.10}$$

with

$$A = \frac{a^2}{d^2}, \qquad a = \frac{1}{d}(d-q)^2, \qquad d = (c^2t^2 - r^2)^{1/2} \tag{A.11}$$

coinciding with the expression (1) by Fock for

$$x^1 = r\cos\varphi\sin\theta\,, \;\; x^2 = r\sin\varphi\sin\theta\,, \;\; x^3 = r\cos\theta\,. \tag{A.12}$$

At the next step we use the equations in variations for the background isotropic solution derived in (Brumberg 1991). The complete metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu\,, \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{A.13}$$

represents the solution of the Einstein field equations

$$R_{\mu\nu} = -\kappa(\mathcal{T}^*_{\mu\nu} + T^*_{\mu\nu}) + \Lambda g_{\mu\nu} \tag{A.14}$$

with

$$\mathcal{T}^*_{\mu\nu} = \mathcal{T}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{T}\,, \qquad T^*_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\,. \tag{A.15}$$

Here $\mathcal{T}_{\mu\nu}$ is the background field mass tensor, $T_{\mu\nu}$ is the perturbation field mass tensor, $R_{\mu\nu}$ is the Ricci tensor, Λ denotes the cosmological constant, $\kappa = 8\pi G/c^2$, and G is the gravitational constant. $\mathcal{T}$ and T are the invariants of the mass tensors $\mathcal{T}^{\mu\nu}$ and $T^{\mu\nu}$, respectively. As usually, each Greek index runs values from 0 to 3, every Latin index runs values from 1 to 3 and the corresponding Einstein summation rule is applied for each twice repeated index.

The background isotropic metric for $\Lambda = 0$, $\mathcal{T}_{\mu\nu} = 0$ reads, in general,

$$\eta_{00} = A\,, \qquad \eta_{0m} = 0\,, \qquad \eta_{mn} = -B\delta_{mn}\,, \tag{A.16}$$

A, B being functions of all four coordinates (in our case $B = A$ and A has the specific form (A.11)).

Equations in variations to determine $h_{\mu\nu}$ are as follows (Brumberg, 1991):

$$\delta R_{\mu\nu} = -\kappa T^*_{\mu\nu} - \kappa\delta\mathcal{T}^*_{\mu\nu} + \Lambda g_{\mu\nu}\,, \tag{A.17}$$

with

$$\delta\mathcal{T}^*_{\mu\nu} = \mathcal{T}^*_{\mu\nu}(g_{\alpha\beta}) - \mathcal{T}^*_{\mu\nu}(\eta_{\alpha\beta}). \tag{A.18}$$

Under coordinate conditions

$$h_{00,0} + h_{ss,0} - 2h_{0s,s} = 0\,, \qquad h_{00,m} - h_{ss,m} + 2h_{ms,s} = 0 \tag{A.19}$$

these equations take the form

$$h_{00,ss} - h_{00,00} = 2L_{00}\,, \tag{A.20}$$

$$h_{0m,ss} = 2L_{0m}\,, \tag{A.21}$$

$$h_{mn,ss} - \frac{B}{A} h_{mn,00} = 2L_{mn} + \Big(\frac{B}{A} - 1\Big) h_{00,mn} - \frac{B}{A}(h_{0m,0n} + h_{0n,0m}) \quad (A.22)$$

with

$$L_{\mu\nu} = B(\kappa T^*_{\mu\nu} + \kappa\delta T^*_{\mu\nu} - \Lambda g_{\mu\nu} + Q_{\mu\nu}), \quad (A.23)$$

$Q_{\mu\nu}$ being non–linear contributions in Ricci tensor components given by (4.3.24)–(4.3.26) of (Brumberg 1991). The use of the background conformally Galilean metric enables one to treat (A.22) as the wave equation with constant coefficients (in contrast to the Robertson–Walker metric in co—moving coordinates with $A = 1$ and B being a function of r and t).

For our purposes it is sufficient to have the simplest, just quasi–Newtonian solution of equations (A.20)–(A.22). First of all, we retain in (A.23) only the first term by putting $L_{\mu\nu} = \kappa B T^*_{\mu\nu}$. We consider only one material point of mass M (the Sun) located at the spatial origin $\mathbf{r} = (x^k) = 0$. Then the contravariant disturbing mass tensor $T^{\mu\nu}$ may be taken in the form

$$T^{\mu\nu} = \frac{\tilde{\rho}}{\sqrt{-g}} \frac{dx^0}{ds} \frac{dx^\mu}{dx^0} \frac{dx^\nu}{dx^0} \quad (A.24)$$

with the density

$$\tilde{\rho} = M\delta(\mathbf{r}) \,, \quad (A.25)$$

$\delta(\mathbf{r})$ being delta–function (Infeld and Plebansky 1960). For the background metric (A.16) one easily finds

$$BT^*_{00} = \frac{1}{2}\sqrt{A}\tilde{\rho}\left(\frac{A}{B} - \frac{v^2}{c^2}\right)^{-1/2}\left(\frac{A}{B} + \frac{v^2}{c^2}\right), \quad (A.26)$$

$$BT^*_{0i} = -\sqrt{A}\tilde{\rho}\left(\frac{A}{B} - \frac{v^2}{c^2}\right)^{-1/2}\frac{v^i}{c}, \quad (A.27)$$

$$BT^*_{ij} = \frac{B}{\sqrt{A}}\tilde{\rho}\left(\frac{A}{B} - \frac{v^2}{c^2}\right)^{-1/2}\left[\frac{1}{2}\left(\frac{A}{B} - \frac{v^2}{c^2}\right)\delta_{ij} + \frac{v^i}{c}\frac{v^j}{c}\right], \quad (A.28)$$

v^i denoting the three–dimensional velocity. In integrating the equations (A.20)–(A.22) for the fixed material point in the conformally Galilean background with $A = B$ one may put

$$L_{00} = 4\pi\sqrt{A}\, m\, \delta(\mathbf{r}) \,, \qquad L_{0i} = 0 \,, \qquad L_{ij} = 4\pi\sqrt{A}\, m\, \delta(\mathbf{r})\delta_{ij} \quad (A.29)$$

with

$$m = \frac{GM}{c^2} \,. \quad (A.30)$$

Hence, by neglecting the retardation terms one can present the approximate solution of (A.20)–(A.22) in form

$$h_{00} = -\frac{2m}{r}\sqrt{A} \,, \qquad h_{0i} = 0 \,, \qquad h_{ij} = -\frac{2m}{r}\sqrt{A}\delta_{ij} \,, \quad (A.31)$$

resulting to expression (4), i.e.

$$ds^2 = \left(A - \frac{2m}{r}\sqrt{A}\right) c^2 dt^2 - \left(A + \frac{2m}{r}\sqrt{A}\right) dx^s dx^s \,. \tag{A.32}$$

It is of interest to compare this expression with the exact solution for the one–body problem in an expanding universe given by McVittie (McVittie 1933; see also Järnefelt 1940, 1942). The McVittie solution for the flat (de Sitter) universe ($k = 0$) has the form

$$ds^2 = \left(\frac{1 - \mu(t)/2r}{1 + \mu(t)/2r}\right)^2 c^2 dt^2 - \left(1 + \frac{\mu(t)}{2r}\right)^4 \mathrm{e}^{\beta(t)} [dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2)] \tag{A.33}$$

with

$$\frac{\dot{\mu}}{\mu} = -\tfrac{1}{2}\dot{\beta}\,, \qquad \mu = m\mathrm{e}^{-\beta(t)/2}\,, \qquad m = \frac{GM}{c^2}. \tag{A.34}$$

Retaining only the first–order terms with respect to m one has

$$ds^2 = \left(1 - \frac{2m}{r}\mathrm{e}^{-\beta/2}\right) c^2 dt^2 - \left(1 + \frac{2m}{r}\mathrm{e}^{-\beta/2}\right) \mathrm{e}^{\beta(t)} [dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2)]. \tag{A.35}$$

Changing the time variable

$$dt = \mathrm{e}^{\beta/2} d\tilde{t} \tag{A.36}$$

one gets the expression

$$ds^2 = \left(\mathrm{e}^{\beta} - \frac{2m}{r}\mathrm{e}^{\beta/2}\right) c^2 d\tilde{t}^2 - \left(\mathrm{e}^{\beta} + \frac{2m}{r}\mathrm{e}^{\beta/2}\right) [dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2)] \tag{A.37}$$

taking for $A = \mathrm{e}^{\beta}$ the same form as (A.32).