XPOZVNP – Veľmi Nezmyselná Prechádzka 16.2.2010

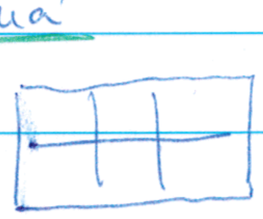
fyzika hľadí na matematiku a títi ju využívať nové veci
fyzika bez matematiky nemôže existovať, má ju ako aparát

ABSTRAKCIA

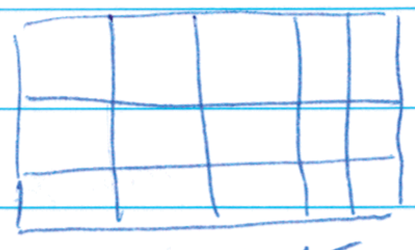
1. číslo

- oddelenie čísla od pojmu (zahrdlenie na 3 banány, 2 hračky, ...)

2. premenná



$3 \cdot 2 = 6$



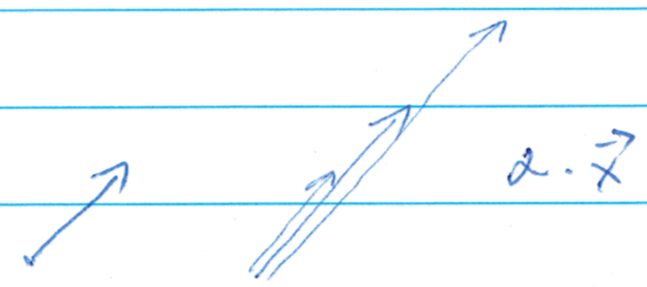
$3 \cdot 5 = 15$

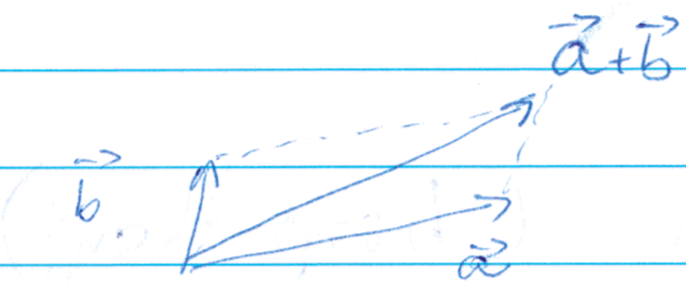
$\rightsquigarrow S = a \cdot b$

$[a]$ = číslo + rozmer ← fyzika do premennej mäša dve veci

3. vlastnosti

- na objektoch nezáleží, zaoberáme sa iba ich vlastnosťami





$\vec{a}+\vec{b} = \vec{b}+\vec{a}$

$\alpha(\vec{a}+\vec{b}) = \alpha\vec{a}+\alpha\vec{b}$

atď.

Tieto operácie začneme vykonávať aj na iných objektoch.

Kvantová mechanika na maticiach } sú ekvivalentné,
funkciách }

na objekte nezáleží, záleží iba na ich vlastnostiach.

Ako sa veci značia – základné elementy typografie matematických objektov

- premenné: a, b, c, d, e (šikmým)

- skratky, číslice, jednotky, ozn. funkcií: stojatým

12 kg t^2 $\sin x$

12 kg e^2 $\mathrm{Sin}\, x$

a_1 a_2 a_3 a_k

a_{max} ω_p

$y = f(x)$
$y = -f(x)$ } tvoriť obe

$\frac{\mathrm{d}\sin x}{\mathrm{d}\, x}$

zložený objekt: tučným rezom písma (bezpatkový)

na tabuli: $\vec{a}$, A

vektor: $\vec{a}$

tenzor: $\overleftrightarrow{a}$

TS - termodynamický systém
TS - teplota . entropia

Rozmery veličín

- fyzikálna veličina vždy veľkosť + rozmer
  - nesabrídať sa jednotky, bez rozmeru to nemá zmysel

$$v = \frac{s}{t} = \frac{6}{3} = 2\ m/s$$

$$1 = m/s \quad ??$$

$$v = \frac{l}{t} = \frac{6m}{3s} = 2\ m/s$$

matematik: $x = A \cdot \sin t$

matematické funkcie nemajú rozmer!

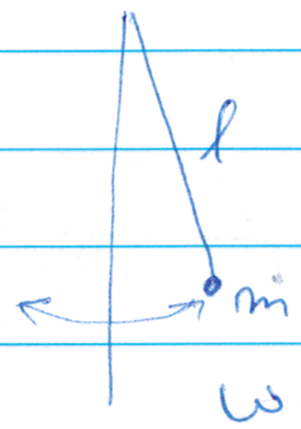
fyzik: $x = A \cdot \sin(\omega t)$

$$[\omega] = \frac{1}{s}$$

$$[t] = s$$

Vždy je dobré kontrolovať rozmery, bezrozmerné musia byť argumenty všetkých matematických funkcií.

Pr. 1:



$$\omega = \omega(l, m, g)$$

$$\omega = l^{\alpha} m^{\beta} g^{\gamma} \ / [\,] \leftarrow \text{skúšame uhádnuť vzťah}$$

rozmerová analýza

rôzne m: hmotnosť m (premenná), meter m

$$\frac{1}{s} = m^{\alpha} kg^{\beta} (m \cdot s^{-2})^{\gamma}$$

$$s^{-1} = m^{\alpha} kg^{\beta} m^{\gamma} \cdot s^{-2\gamma}$$

$s: -1 = -2\gamma$, $m: 0 = \alpha + \gamma$, $kg: 0 = \beta$ $\Rightarrow$ $\beta = 0$, $\gamma = \frac{1}{2}$, $\alpha = -\frac{1}{2}$

Z toho hneď vidíme, že:

$$\omega = l^{\alpha} m^{\beta} g^{\gamma}$$

$$\omega = l^{-1/2} g^{1/2} = \sqrt{\frac{g}{l}}$$

$$\boxed{\omega = \sqrt{\frac{g}{l}}}$$

Malý podrobí: Z rozmerovej analýzy vieme iba

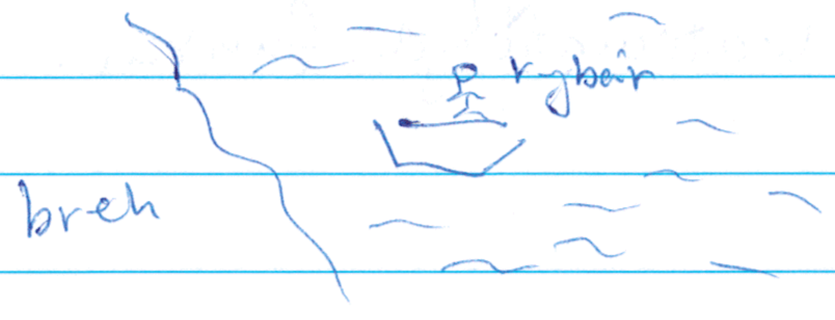
$$\omega \sim \sqrt{\frac{g}{l}}$$

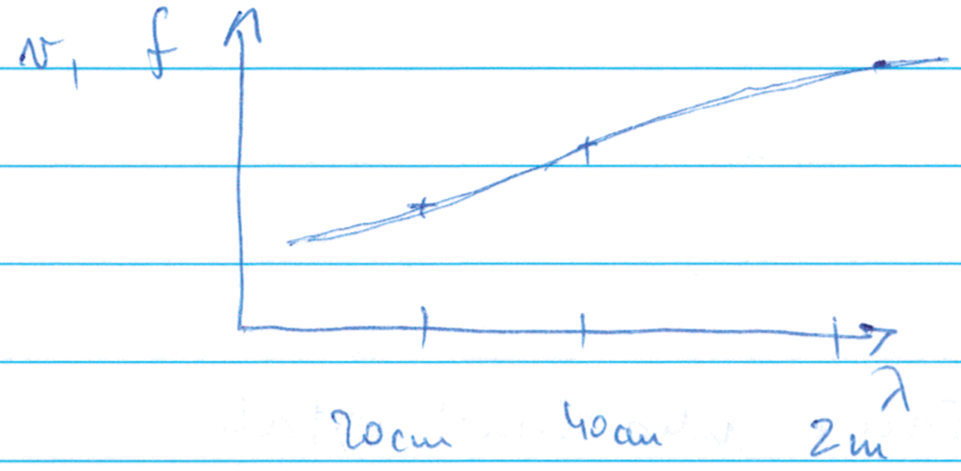
Mohlo by platiť aj $\omega = 2\sqrt{\frac{g}{l}}$

$\omega = \pi\sqrt{\frac{g}{l}}$

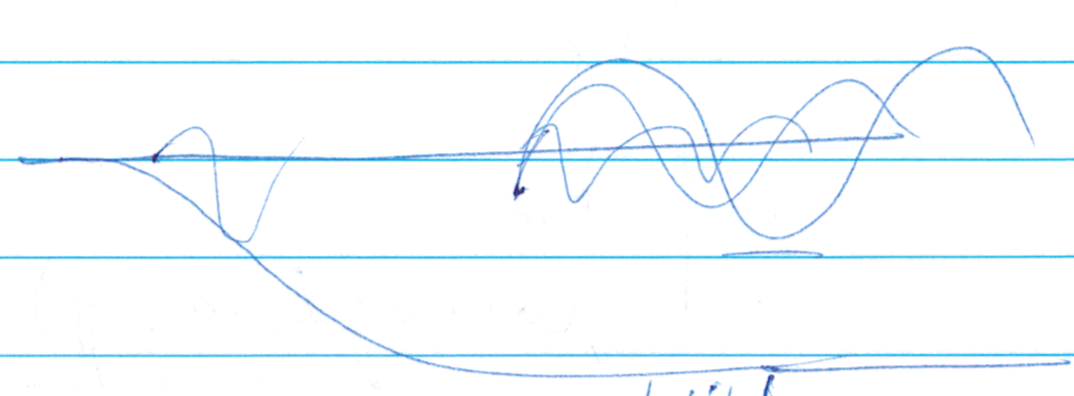
Teraz nám však stačí jeden experiment na to, aby sme tento číselný koeficient zistili.

Pr. 2.:





$v = v(\lambda, g, \varphi)$



na hĺbke závisí iba pri brehu, kde sa dotýka dna

$v = \lambda^\alpha g^\beta \rho^\gamma \;/[\,]$ – rozmerová analýza rýchlosti v/n na tvori

$m \cdot s^{-1} = m^\alpha (m \cdot s^{-2})^\beta (kg \cdot m^{-3})^\gamma$

$m \cdot s^{-1} = m^\alpha m^\beta \cdot s^{-2\beta} kg^\gamma m^{-3\gamma}$

$m: \quad 1 = \alpha + \beta - 3\gamma$

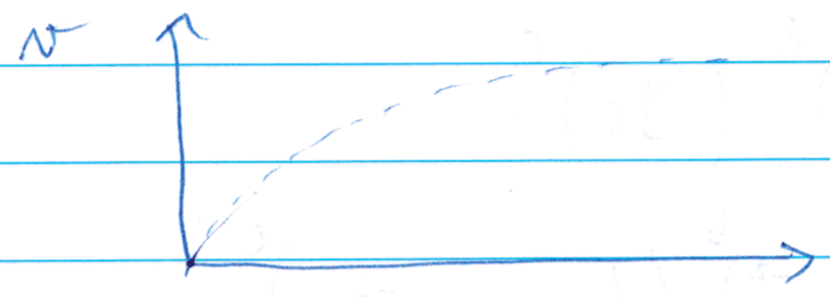
$s: \quad -1 = -2\beta$

$kg: \quad 0 = \gamma$

$\gamma = 0 \;;\; \beta = \frac{1}{2} \;;\; \alpha = 1/2$

$v = \lambda^\alpha g^\beta \rho^\gamma = \lambda^{1/2} g^{1/2} = \sqrt{\lambda g}$

teda $\boxed{v \sim \sqrt{\lambda g}}$



$\omega = \sqrt{g k}$

$v = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$ ← odvodené teoreticky

Pr. 3.:

Konštanta – rýchlosť šírenia svetla : $C = 3 \times 10^8 \; m \cdot s^{-1}$

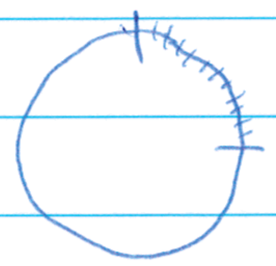
Špeciálna relativita – oprava mechaniky aby bola v súlade s elektrodyn.

$G = 6{,}67 \times 10^{-11} \; N m^2 kg^{-2}$ – gravitácia

$F = G \frac{m_1 m_2}{r^2}$

Planckova konšt.: $\hbar = 1{,}05 \times 10^{-34} \; J \cdot s$ (mikrosvet)

Meter:



10 000 000

pohodník prechádzajúci Parížom

Prirodzené jednotky:

Planckov čas: $t_p = c^\alpha G^\beta \hbar^\gamma$

dĺžka $l_p = c^a G^b \hbar^\gamma \quad / [\,]$

$E_p = \dots$

$m_p = \dots$

$$m^1 = (m \cdot s^{-1})^a (N m^2 kg^{-2})^b (Js)^c$$

$$m^1 = m^a \cdot s^{-a} \cdot \underbrace{(kg \cdot m \cdot s^{-2} m^2 kg^{-2})^b}_{N} (kg \cdot m^2 s^{-2} s)^c$$

$$m^1 = m^a \cdot s^{-a} kg^b m^b s^{-2b} m^{2b} kg^{-2b} kg^c m^{2c} s^{-2c} s^c$$

$m: 1 = a + b + 2b + 2c$

$s: 0 = -a - 2b - 2c + c$

$kg: 0 = b - 2b + c \quad \Rightarrow \quad b = c$

$$\left.\begin{aligned} 1 &= a + 5c \\ 0 &= -a - 3b \end{aligned}\right\} +$$

$$1 = 2b \quad \Rightarrow \quad b = \frac{1}{2}$$

$$c = \frac{1}{2}$$

$$a = -3b = -\frac{3}{2}$$

$$l_p = c^{-3/2} G^{1/2} \cdot \hbar^{1/2}$$

$$l_p = \sqrt{\frac{G\hbar}{c^3}} = 10^{-35}\ m$$

My dokážeme: 10^{-18} m, potom máme nestačia mikroskopy

$$v = \frac{l}{t} = \frac{6\,\text{km}}{3\,\text{h}} = 2\,\text{m/s}$$

chýbajú jednotky

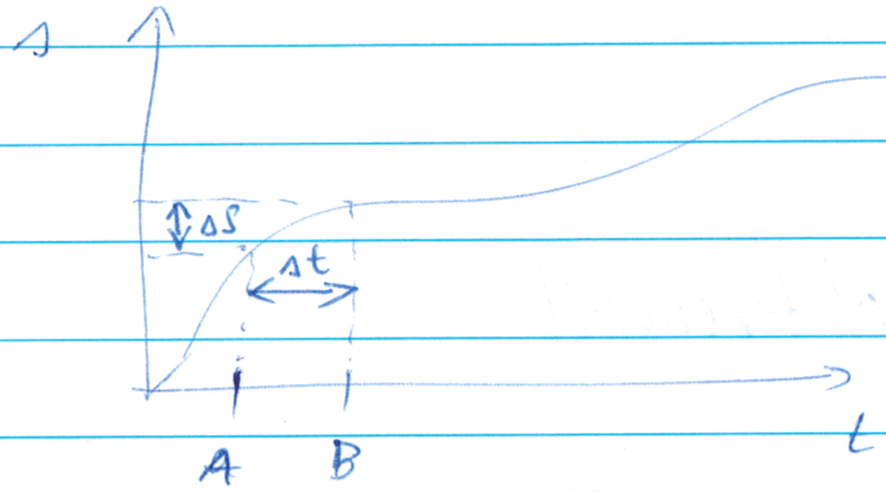
$$L = \log \frac{I}{I_0} = \log I - \log I_0$$

↳ argumenty tu nemáme byť!

$$\log \frac{10A}{20A} \qquad \log 10A = \log 10 + \log A$$

↳ Čo to je? Problém!

Derivácie



$$\bar{v} = \frac{s_B - s_A}{t_B - t_A} = \frac{\Delta s}{\Delta t}$$ priemerná rýchlosť

$$\bar{v} = \frac{s(t_A + \Delta t) - s(t_A)}{\Delta t}$$

$t_A + \Delta t = t_B$

[1] $$v = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}$$ ← zápis matematika

[2] $$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$ ← zápis fyzika

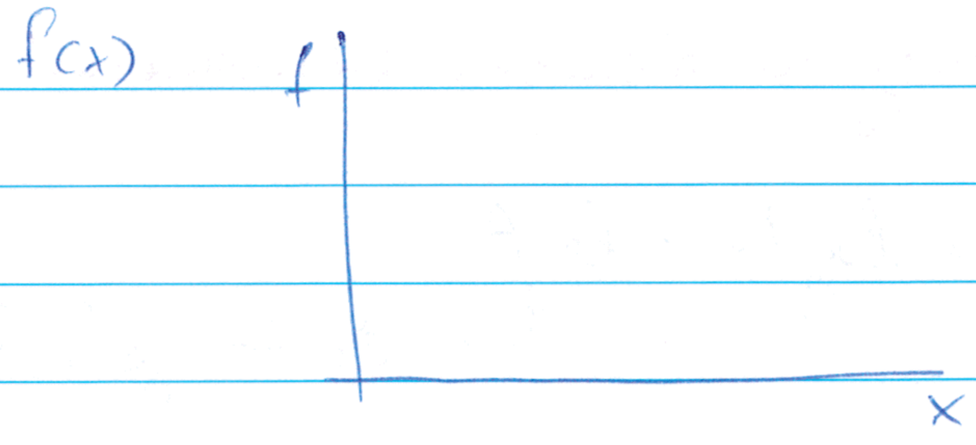
Δ - konečný prírastok
d - nekonečne malý prírastok (diferenciál)
δ - jeden z možných prírastkov (variácia)

BOBO

[3] $v = \frac{ds}{dt}$ – limita je schovaná práve v d

Je to stále to isté, iba sa snažíme zjednodušiť zápis.

Je mnoho možností ako sa dostať do okolia



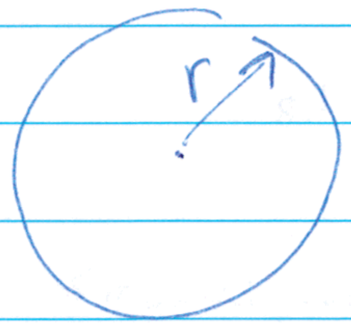
$$f' \equiv \lim \frac{\Delta f}{\Delta x}$$

$$f' = \frac{df}{dx}$$

$$\boxed{df = f' \cdot dx}$$ 1. diferenciál

$$\boxed{\Delta f \doteq f' \cdot \Delta x}$$ Lagrangeova veta o prírastku

Pr.:

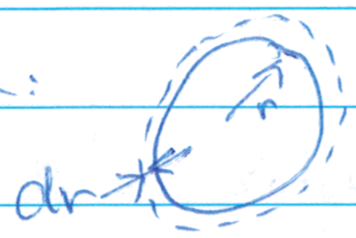


$S(r) = \pi r^2$

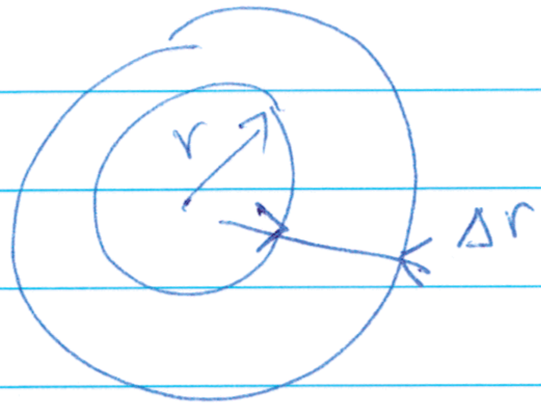
$dS = S' dr$

$dS = 2\pi r \, dr$

pridám:



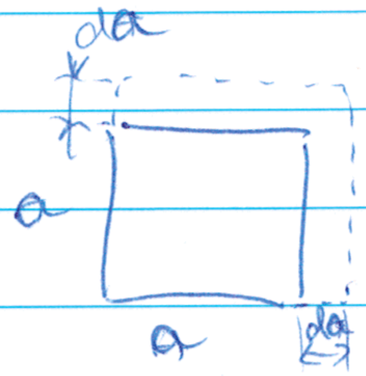
dS: $\doteq \quad 2\pi r \, dr$



$\Delta S \overset{?}{=} 2\pi r \Delta r$

Neplatí - to čo pridám už neviem narovnať do obdĺžnika.

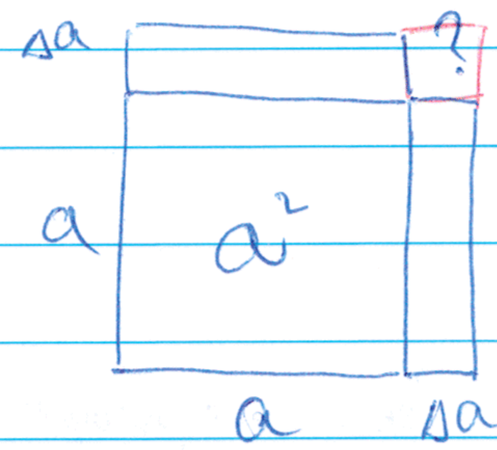
$$\Delta S \doteq 2\pi r \Delta r \quad \text{(približne)}$$



$S(a) = a^2$

$dS = S' \cdot da$

$dS = 2a \cdot da$



$\Delta S \doteq 2a \Delta a$

$f(x_1, x_2, x_3, \dots, x_N)$

$$df = \frac{\partial f}{\partial x_1} \cdot dx_1 + \frac{\partial f}{\partial x_2} \cdot dx_2 + \dots + \frac{\partial f}{\partial x_N} dx_N \leftarrow$$

parciálna derivácia je ako každá iná, iba máme možnosť vybrať podľa čoho derivujem

1. diferenciál

$$\Delta f \doteq \frac{\partial f}{\partial x_1} \cdot \Delta x_1 + \frac{\partial f}{\partial x_2} \cdot \Delta x_2 + \dots + \frac{\partial f}{\partial x_N} \cdot \Delta x_N$$

Lagrangeova v.

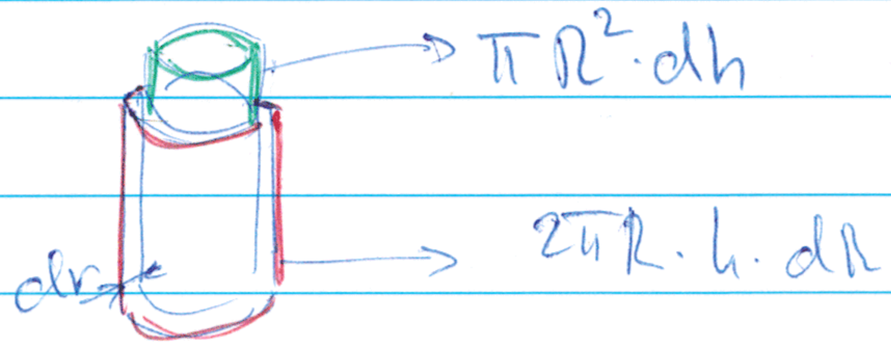
Př.:

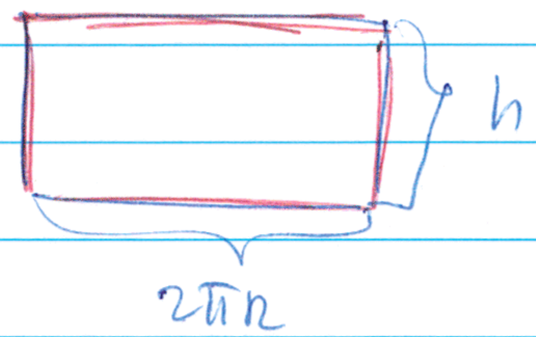
$$V(R, h) = 2\pi R^2 \cdot h$$

$$dV = \frac{\partial V}{\partial R} \cdot dR + \frac{\partial V}{\partial h} \cdot dh$$

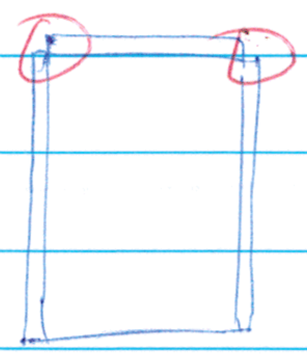
$$\boxed{dV = 2\pi R h \cdot dR + \pi R^2 dh}$$

1. diferenciál objemu válce



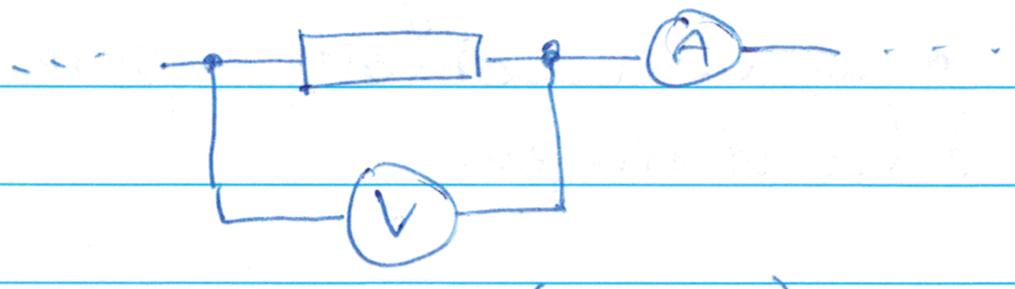


$$\Delta V \doteq 2\pi R h \, \Delta R + \pi R^2 \Delta h$$



– roby sou sem nepridal

Př. 3:



$$I = (2 \pm 0{,}1)\ A$$
$$U = (10 \pm 0{,}2)\ V$$

$$R = \frac{U}{I} = \frac{10V}{2A} = 5\,V/A = \underline{5\,\Omega}$$

Ako presne sme zmeral odpor?

$$R(U, I) = \frac{U}{I}$$

$$dR = \frac{\partial R}{\partial U} \cdot dU + \frac{\partial R}{\partial I} \cdot dI$$

$$dR = \frac{1}{I} \cdot dU - \frac{U}{I^2} dI$$

$$\Delta R \doteq \underbrace{\frac{1}{2} \cdot (\pm 0{,}2)\,\Omega}_{\text{chyba voltmetra}} - \underbrace{\frac{10}{4} (\pm 0{,}1)\,\Omega}_{\text{chyba ampérmetra}}$$

$$\Delta R \doteq \pm 0{,}1\,\Omega \pm 0{,}25\,\Omega$$

$$\Delta R_{max} \doteq \left|\frac{\partial R}{\partial U} \Delta U\right| + \left|\frac{\partial R}{\partial I} \Delta I\right| = 0{,}1\,\Omega + 0{,}25\,\Omega = \underline{0{,}35\,\Omega}$$

$R = (5 \pm 0{,}35)\,\Omega$ – uvažujeme najväčšiu chybu ktorej sme sa mohli dopustiť

diferenciál – nekonečne malé prírastky, presný vzťah
Lagrangeova veta – konečné prírastky, približný vzťah

Pr.: $f(x, y, z) = x^2 y \sin z$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$df = 2xy \sin z + x^2 \sin z + x^2 y \cos z\, dz$$ úplný diferenciál

$$df = 2xy \sin z + x^2 \sin z$$ nie je úplný diferenciál

BOBO

Einsteinova sumačná konvencia

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots + \frac{\partial f}{\partial x_k} dx_k$$

$$= \sum_{\xi=1}^{N} \frac{\partial f}{\partial x_\xi} dx_\xi$$

$$= \frac{\partial f}{\partial x_\xi} dx_\xi$$

Pr.:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 \cdot b_2 + \dots + a_k b_k$$

$$= \sum_{k=1}^{N} a_k b_k$$

$$= a_k b_k$$

$$a_k b_k = a_1 b_1 + a_2 \cdot b_2 + \dots$$

$$a_l b_l = a_1 b_1 + a_2 b_2 + \dots$$

↳ táto suma sa robí cez všetky možné hodnoty (býva zrejmé z kontextu)

Pr.: $\operatorname{div} \vec{K} = \frac{\partial K_1}{\partial x_1} + \frac{\partial K_2}{\partial x_2} + \frac{\partial K_3}{\partial x_3} = \sum_i \frac{\partial K_i}{\partial x_i} = \frac{\partial K_i}{\partial x_i}$

$$f(x) = \sum C_k f_k = C_k f_k$$

Pr.: $C_{kl} = A_{km} B_{ml}$ – čo sa deje s m?

$$= \sum_{m=1}^{\dots} A_{km} B_{ml}$$ – násobenie matíc $C = A \cdot B$

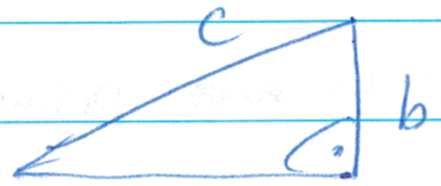
$$C_{kl} = A_{km} B_{ml}$$

m – sčítací index (nemý index)

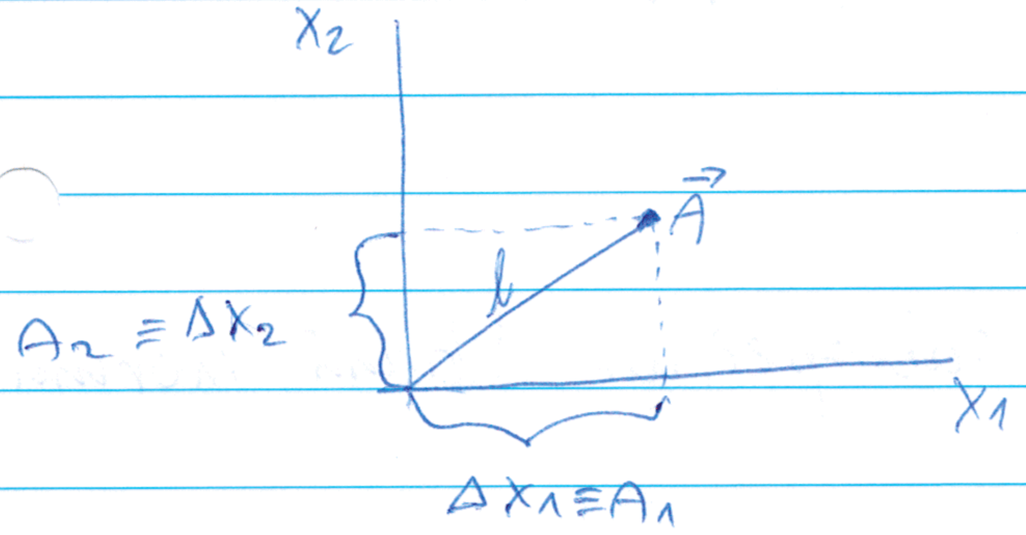
k, l – volné indexy, ces ne sa nesčítá,

musia byť na oboch stranách rovnosti

Pythagorova veta



$$c^2 = a^2 + b^2$$



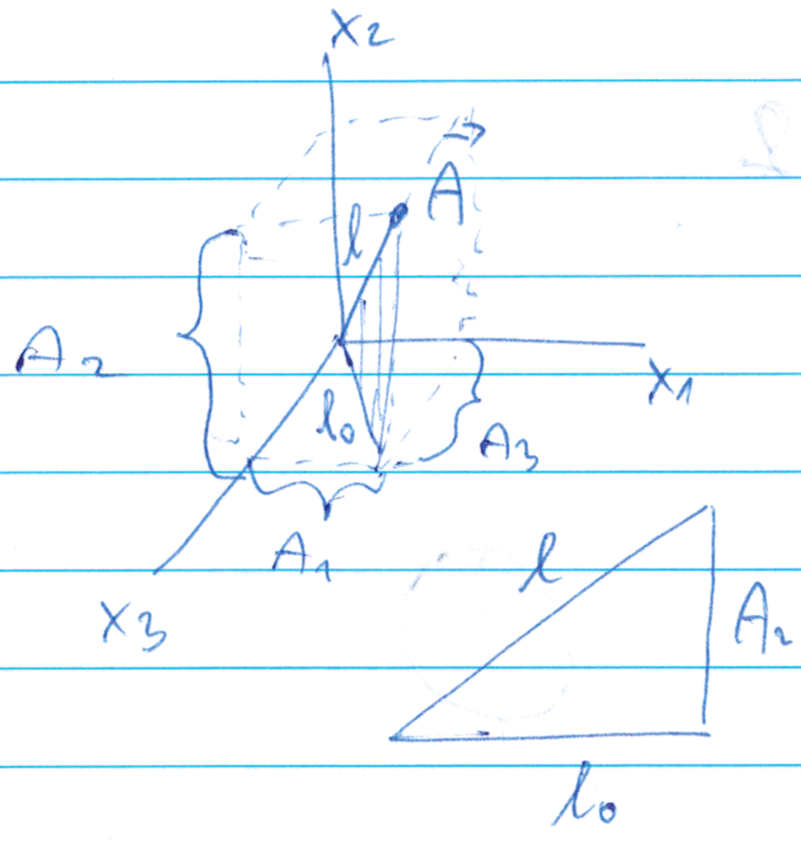
l – dĺžka vektora

$$l^2 = (\Delta X_1)^2 + (\Delta X_2)^2$$

$$l^2 = A_1^2 + A_2^2$$

$$l = \sqrt{A_1^2 + A_2^2}$$

v 3D:



Chceme ukázať:

$$l_0^2 = A_1^2 + A_3^2$$

$$l^2 = l_0^2 + A_2^2 \Rightarrow l = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

Obecne v N-dimenziách platí, že

$$l = \sqrt{A_1^2 + A_2^2 + \cdots + A_N^2} = \sqrt{A_{kk}}$$

BOBO

$$l = \left(A_1^2 + A_2^2 + \dots\right)^{1/2}$$

$$l = \|A\|$$

↳ veľkosť vektom (norma)

Prečo práve na 1/2? Otvaríme si to písmenami:

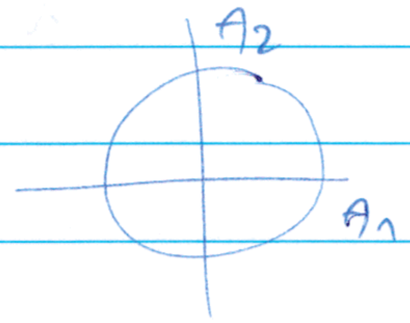
$$l = \|A\| = \left(|A_1|^p + |A_2|^p + \dots + |A_k|^p\right)^{1/q}$$

p, q vzdialenosť

absolútnou hodnotou zabezpečíme kladnú normu

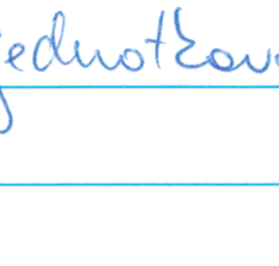
Definícia kružnice:

$K = \{A;\ \|A\| = 1\}$ (jednotkové)



Euklidov priestor: $p = 2,\ q = 2$

$$\|A\| = 1$$
$$\sqrt{A_1^2 + A_2^2} = 1$$
$$\sqrt{x^2 + y^2} = 1$$
$$\boxed{x^2 + y^2 = 1}$$



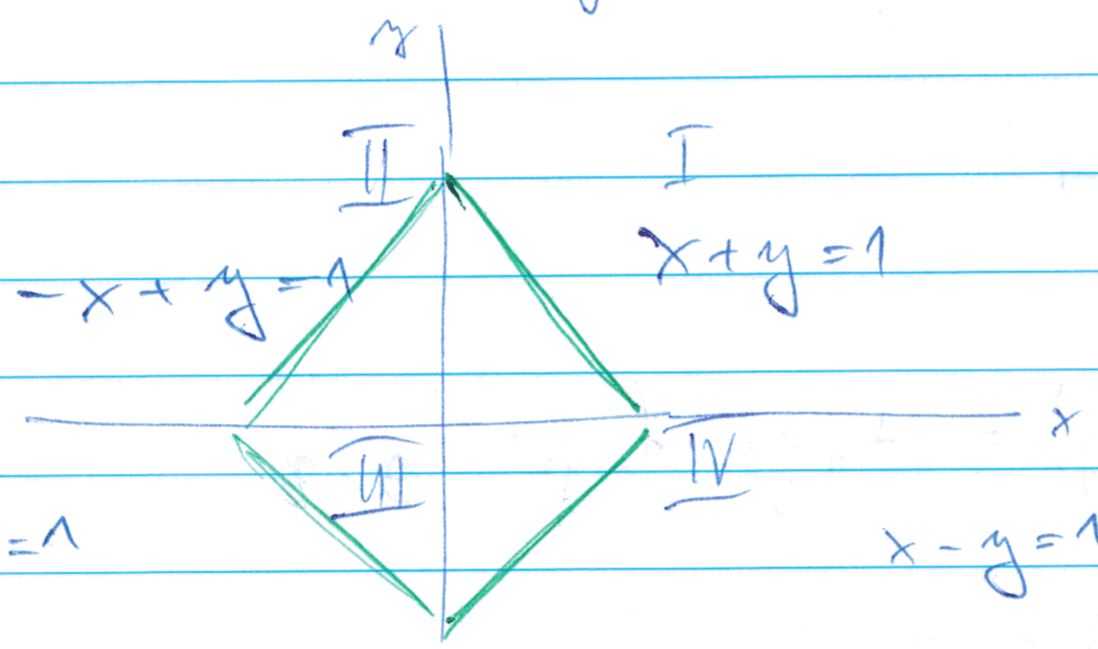
~~$p = 1,\ q = 1$~~ obecne v p, q priestore:

$$\|A\| = 1$$
$$\left(|A_1|^p + |A_2|^p\right)^{1/q} = 1$$
$$\left(|x|^p + |y|^p\right)^{1/q} = 1$$

ak $p=2$, $q=2$:

ak $p=1$, $q=1$:

$$|x| + |y| = 1$$



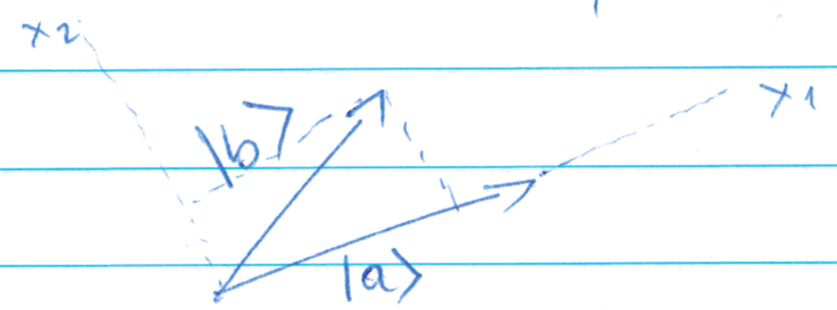
↳ je to stále kružnica, ale s inou normou

Skúsiť si vykresliť pre rôzne p, q.

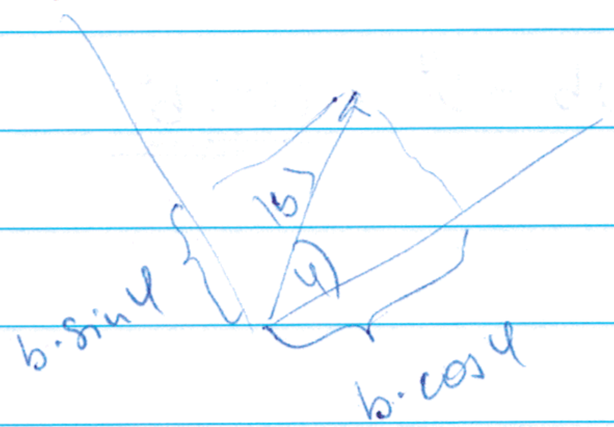
Je to dobré pre zakrivené časopriestory. Neskôr to uvidíme lepšie.

Skalárny súčin

$\vec{a}, \vec{b}$; $\vec{a} \cdot \vec{b} = a_i b_i = a_1 b_1 + a_2 b_2 + \dots$



$|a\rangle = (a, 0)$



$|b\rangle = (b \cdot \cos\varphi, b \cdot \sin\varphi)$

skal. súčin: $\vec{a} \cdot \vec{b} = \langle a|b\rangle = a_1 b_1 + a_2 b_2 = ab\cos\varphi + 0$

$$\boxed{\langle a|b\rangle = ab \cdot \cos\varphi}$$

$$\cos\varphi = \frac{\langle a|b\rangle}{a\,b} = \frac{\langle a|b\rangle}{\|a\|\,\|b\|}$$

$$\|a\| = \sqrt{a_1^2 + a_2^2 + \dots + a_N^2} =$$

$$= \sqrt{\vec{a}\cdot\vec{a}}$$

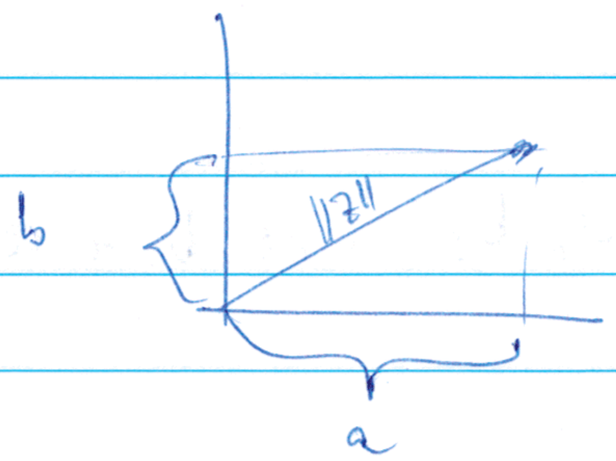
$$\|a\| = \sqrt{\langle a|a\rangle}$$

na $\mathbb{R}^3$: $\langle f|g\rangle = f_1 g_1 + f_2 g_2 + f_3 g_3$

$\mathbb{R}^N$: $\langle f|g\rangle = f_k g_k$

$\mathbb{C}^N$:

$z = a + ib$



$$\|z\| = \sqrt{a^2 + b^2}$$

$$z\cdot z^* = (a+ib)(a-ib) = a^2 + iba - iab + b^2 = a^2 + b^2$$

$$\|z\| = \sqrt{z\cdot z^*}$$

$$\|f\| = \sqrt{f\cdot f^*}$$

$$\langle f|g\rangle = f_1 g_1^* + f_2 g_2^* + \dots + f_N g_N^*$$

$$\langle f|f\rangle = f_1 f_1^* + f_2 f_2^* + \dots + f_N f_N^*$$

komplexné združenie zabezpečí, že

$$f_1 f_1^* \geq 0$$

- vieme dokonca zvoliť, že komplexne združenie je ten druhý

priestor l^2: $\{f_1, \dots, f_N, \dots\}$

$$\langle f|g\rangle = f_1 g_1^* + f_2 g_2^* + \dots$$

$$= \sum_{i=1}^{\infty} f_i g_i^*$$

uhol medzi dvoma postupnosťami:

$$\cos\varphi = \frac{\langle f|g\rangle}{\|f\| \cdot \|g\|} =$$

$$= \frac{\sum f_i g_i^*}{\sqrt{f_i f_i^*}\ \sqrt{g_i g_i^*}}$$

Spojité indexy:

L^2 $\quad f_x \quad x \in \mathbb{R}$

historicky:

$f_x \quad x = \{1, 2, 3, \dots\}$

$f(x) \quad x \in \mathbb{R}$

$\Rightarrow$ spojitým indexom rozumieme funkcie

$$\langle f(x) | g(x) \rangle = \int f(x)\, g^*(x)\, dx$$

$$\cos\varphi = \frac{\langle \exp x | x^5 \rangle}{\| \exp x \| \, \| x^5 \|}$$