Printed in U.S.A.

Paper No. 72-Aut-N

an ASME publication

G. F. OSTER

D. M. AUSLANDER

Mechanical Engineering Department, University of California, Berkeley, Calif.

The Memristor: A New Bond Graph Element

The "memristor," first defined by L. Chua for electrical circuits, is proposed as a new bond graph element, on an equal footing with R, L, & C, and having some unique modelling capabilities for nonlinear systems.

The Missing Constitutive Relation

In his original lecture notes introducing the bond graph technique, Paynter drew a "tetrahedron of state," (Fig. 1) which summarized the relationship between the state variables (e, f, p, q) [1, 2].[1] There are 6 binary relationships possible between these 4 state variables. Two of these are definitions: the displacement, $q(t) = q(0) + \int_0^t f(t)dt$ and the quantity $p(t) = p(0) + \int_0^t e(t)dt$, which is interpreted as momentum, magnetic flux, or "pressure-momentum" [2]. Of the remaining four possible relations, three are the elementary constitutive relations for the energy storage and dissipation elements:

$$F_C(e, q) = 0 \tag{1a}$$

$$F_I(p, f) = 0 \tag{1b}$$

$$F_R(e, f) = 0 \tag{1c}$$

What of the missing constitutive relation (which Paynter draws as a "hidden line" in Fig. 1)? From a purely logical viewpoint, this constitutive relation is as "fundamental" as the other three!

Recently, L. Chua pointed out that we may have been too hidebound in our physical interpretations of the dynamical variables [3]. After all, they are only mathematical definitions. He proposed that the missing constitutive relation,

$$F_m(q, p) = 0 \tag{2}$$

be called a "memristor," i.e., memory resistor, since it "remembers" both integrated flow and total applied effort.

What distinguishes a memristor from the other basic elements? What are its properties, and what effects, if any, does it model? Chua found few applications within the confines of electrical circuit theory; however, if we look beyond the electrical domain it is not hard to find systems whose characteristics are conveniently represented by a memristor model.

Properties

The constitutive relation for a 1-port memristor is a curve in the q-p plane, Fig. 2. In this context, we do not necessarily interpret the quantity $p(t) = p(0) + \int_0^t e(t)dt$ as momentum, flux, or pressure-momentum (2), but merely as the integrated effort ("impulse").

Depending on whether the memristor is charge- or impulse-controlled we may express the constitutive relation as

$$q = F(p) \quad \text{impulse-controlled} \tag{3a}$$

$$p = G(q) \quad \text{charge-controlled} \tag{3b}$$

Differentiating, we obtain

$$\dot{q} = F'(p)\ \dot{p} \text{ or } f = W(p)e \tag{4a}$$

$$\dot{p} = G'(q)\ \dot{q} \text{ or } e = M(q)f \tag{4b}$$

where $M(q)$ is called the incremental "memristance" and $W(p)$

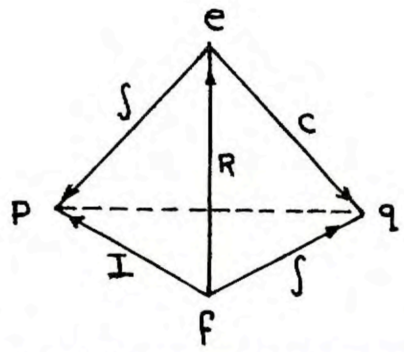

Fig. 1 Relation of state variables and constitutive relations ("Tetrahedron of State," Paynter, 1961)

[1] Numbers in brackets designate References at end of paper.

Contributed by the Automatic Control Division for publication (without presentation) in the Journal of Dynamic Systems, Measurement, and Control. Manuscript received at ASME Headquarters, April 28, 1972. Paper No. 72-Aut-N.

Copies will be available until September, 1973.

Discussion on this paper will be accepted at ASME Headquarters until January 2, 1973

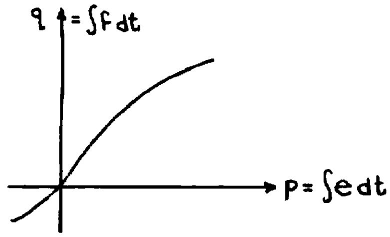

Fig. 2 Memristor constitutive relation

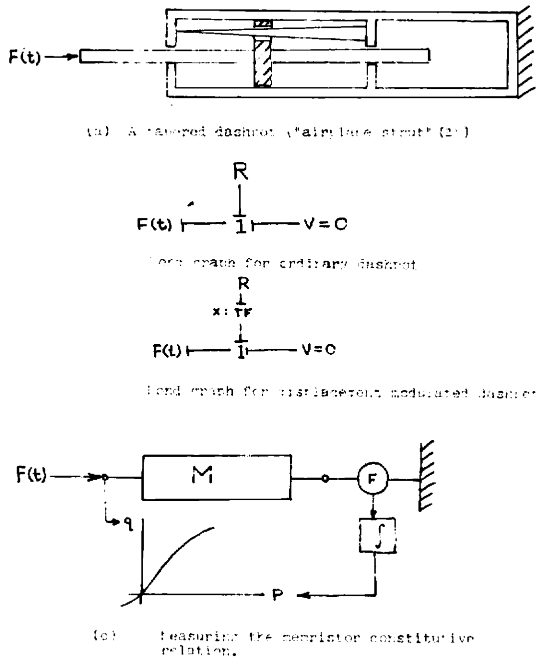

Fig. 3

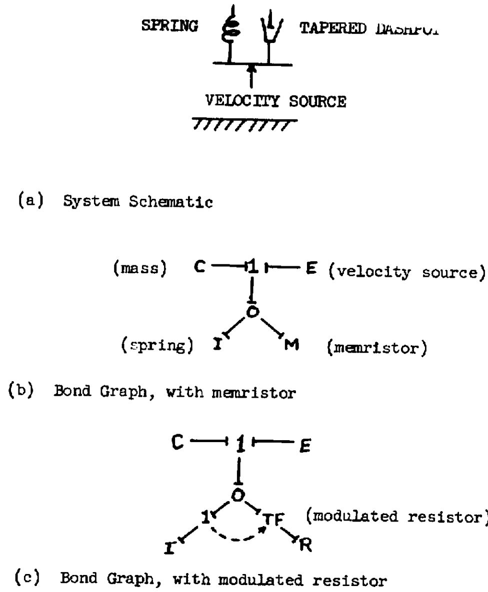

Fig. 4 Schematic and bond graph of mechanical system with displacement-modulated dashpot

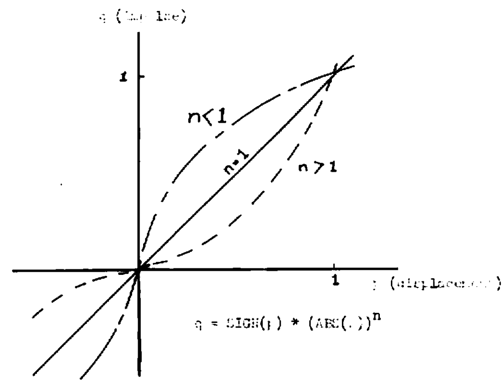

Fig. 5 Memductance curves

the incremental "memductance." We see that, dynamically, the memristor appears as either an impulse or a charge modulated resistor. Notice that for the special case of a linear constitutive relation, M = constant and W = constant, a memristor appears as an ordinary resistor. So memristors have meaning only for nonlinear systems (which may account in part for their neglect till now). Furthermore, a glance at the "tetrahedron of state" Fig. 1, shows that, since both an integration and a differentiation are involved in viewing the memristor as a "resistor" (i.e., on the e-f plane), the memristor, like the resistor, is causally neutral. That is, it may accept either an effort or a flow as input variable. However, there appear to be some restrictions insofar as device modeling is concerned which will be mentioned in the following.

Since the memristor is a 1-port device, it is trivially reciprocal in (q-p) coordinates. However, it is obvious that all definitions may be easily extended to the case of multiport, nonreciprocal resistors [3].

What then distinguishes a memristor from a resistor? Consider, for example, the tapered dashpot shown in Fig. 3. If we attempt to characterize this device on the e-f plane, mistaking it for a true resistor, we would not obtain a unique constitutive relation, $F(e, f) = 0$, but rather some peculiar hysteretic behavior, since the incremental resistance depends on the instantaneous piston displacement. On first glance one might attempt to model this device with a modulated R, so that the resistor constitutive relation could be parameterized by the state variable x, Fig. 3(b).[2] However, x is not a defined state variable for any element in the system. What is required is the displacement of the dashpot itself.[3] Modeling this device as a memristor eliminates the cumbersome modulation, and permits us to characterize the device as a *single* curve in the x-p plane. An important restriction, which is apparent from the form of the constitutive relation, is that the memristor can only be used to model linear, displacement modulated resistors. A real dashpot, for example, might have a characteristic like $e = F(q)f|f|$. The experimental setup to measure the constitutive relation for the tapered dashpot is shown in Fig. 3(c).

Examples

We have simulated both mechanical and electrochemical systems with memristors. The mechanical system, which includes a tapered dashpot of the type described in the foregoing, might

[2]Note: We are using the effort:velocity, flow:force analogy.

[3]An example of a true displacement modulated resistor is an electrolytic solution, where the number of charge carriers may vary with the electrolyte concentration [5]. Such concentration modulation is always implicitly present in electrical systems since the definition of the flow variable contains a concentration term which is included in the resistance: $f = qv_d$, where v_d = electron or ion drift velocity [4].

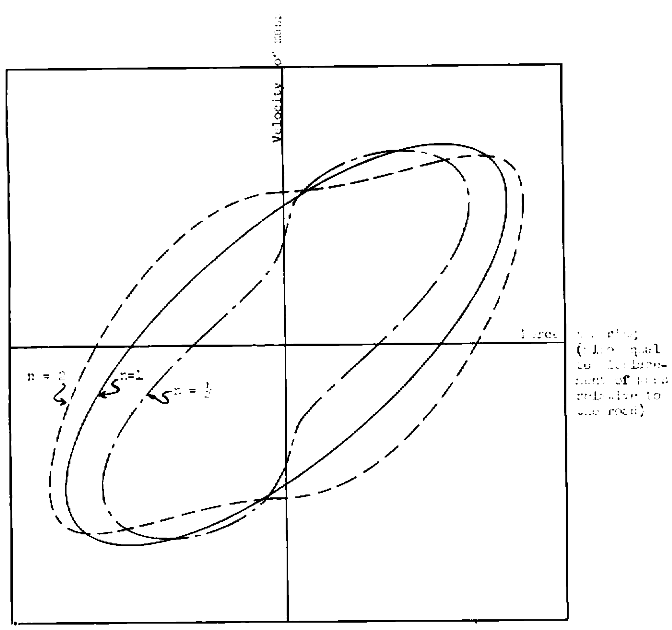

Fig. 6 Sinusoidal response on the state-plane

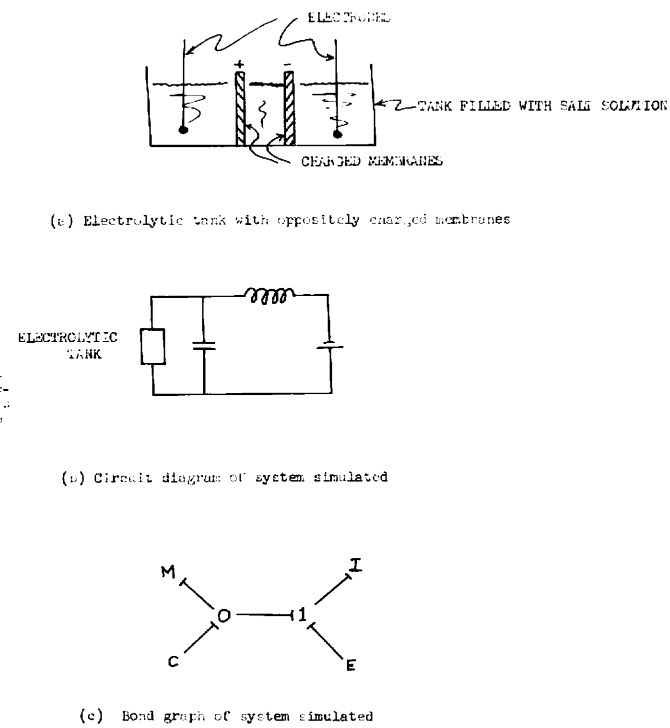

Fig. 7 Electro-chemical memristor system

be considered a crude model of an automobile suspension using a shock absorber whose characteristics depend on displacement. The electro-chemical system is a simple circuit containing a membrane rectifier [5]: an electrolytic cell whose electrical resistance depends on the electrolyte concentration between two charged membranes.

The schematic of the mechanical system is shown in Fig. 4(*a*). The mass could represent the mass of the car, the spring and dashpot its suspension system, and the velocity source the input due to undulations in the road. The bond graph, with the tapered dashpot as a memristor, is shown in Fig. 4(*b*) (using the e = velocity, f = force definitions). It is an accident of this particular system that it is also possible to represent the tapered dashpot as a modulated resistor since the displacement of the dashpot is the same as the displacement of the spring and thus proportional to the force in the spring. The bond graph for this system is shown in Fig. 4(*c*). Note that even in this case which has a standard bond graph representation, the bond graph with the memristor uses fewer elements and avoids the necessity of defining an entirely different kind of bond, the dashed bond for modulation.

Power-law relations used for the memductance are shown in Fig. 5 for the three cases that were simulated. Since, as was shown in the foregoing, the conductance depends on the slope of the memductance curves, the curve marked $n > 1$ corresponds to a dashpot that has a monotone increasing conductance. This is reversed for the curve marked $n < 1$, and for $n = 1$ the conductance is the same everywhere.

The response of the system to a sinusoidal forcing velocity is shown in the state plane in Fig. 6. The curves are coded in the same manner as used in Fig. 5; the solid line is for $n = 1$, the dashed line for $n = 2$, and the dash-dot line for $n = 1/2$. These results were consistent with those computed using the modulated resistor instead of the memristor. The case $n = 1$ corresponds to an ordinary linear dashpot, and, as expected, its state-plane trajectory is an ellipse. The other two trajectories are non-elliptical, indicating that the nonlinear p-q relation caused frequencies other than the forcing frequency to appear in the output. The presence of these harmonics is typical of nonlinear systems. The slopes of the trajectories near the velocity axis (that is, for spring-force close to zero) shows the characteristics of the displacement modulated dashpot; for $n = 2$ the dashpot is very soft around its center and the trajectory shows this by being nearly horizontal. On the other hand, for $n = 1/2$, the dashpot is very stiff near the center and its trajectory is very steep. (In theory, for $n = 1/2$ the slope is infinite at the origin. Use of a finite-difference solution, however, replaces the infinite slope with a large, but finite, slope near the origin, a much more realistic situation.)

In the system shown in Fig. 7(*a*) two oppositely charged membranes are introduced into a tank with two electrodes as shown. Because the oppositely charged membranes selectively prevent the passage of co-ions (i.e., ions with the same charge as the membranes), when an electric current flows through the electrodes the net electrolyte concentration in the inter-membrane space will increase or decrease, depending on the direction of current flow. Since the apparent electrical conductivity goes down as the concentration of ions decreases, the resistance of this device, as viewed from the external circuit, will depend on the total amount of current that has flowed through the cell. The concentration, and thus the resistance, will continue to change as long as there is any current flow.[4]

We have simulated the system shown in Fig. 7(*b*); its bond graph is shown in Fig. 7(*c*). In this case there are no variables anywhere in the system that could be used to provide the modulation for a modulated resistance. Thus, the memristor is the only possible element that can be used to model the electrolytic tank, short of a full-scale model of the ionic flows [5]. Note that, although the memristor appears as a dissipative element, it is a dynamic device requiring the independent specification of an initial condition, $q(0)$. The state space for the system of Fig. 7(*b*) is 3-dimensional, not 2-dimensional as would be expected on the basis of an RLC model.

Since concentrations can never be negative, we expect an asymmetric constitutive relation in the p-q plane; if we assume

[4] In the model simulated here it is assumed that the membranes are perfectly selective. In the actual case a steady-state can be established at very high (or very low) concentrations because the gradient for diffusion becomes high enough so that there will be some flow of the co-ions through the membranes, thus stabilizing the intermembrane concentration [5].

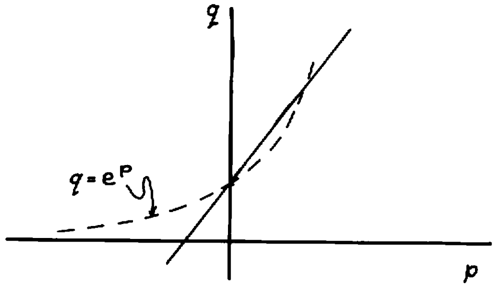

Fig. 8 Memristance curves

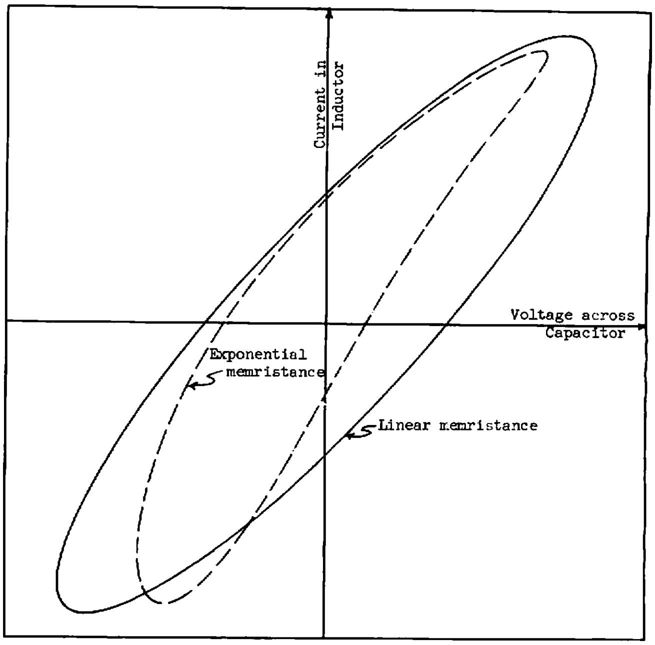

Fig. 9 State-plane trajectories for sinusoidal excitation

that the electrical resistance is inversely proportional to the concentration in the intermembrane space one obtains an exponential memristance curve, as shown by the dashed line in Fig. 8. The linear memristance shown by the solid line yields a completely linear system (i.e., constant resistance) for comparison purposes.

The results on the state plane projection are shown in Fig. 9 for sinusoidal excitation. As expected, the linear memristance (solid line) has an elliptical trajectory indicating the absence of any harmonics. The nonlinear case shows behavior indicative of an output containing more than just the forcing frequency and, because of the asymmetric constitutive relation of the memristor, its trajectory is also asymmetric.

There are two apparent restrictions on the use of the memristor as a modeling device. (i) According to equation (4), the memristor, viewed on the e-f plane, models a *linear* displacement-modulated resistor. That is, the device appears nonlinear by virtue of the nonlinear p-q relationship; but the $f = f(e, q)$ surface representing the resistor characteristic can only be a ruled surface, i.e., a surface swept out by a straight line. (ii) Memristors whose constitutive relation becomes horizontal (q = constant) are vertical (p = constant) are usually not admissible models, since the device continues to integrate the effort even though displacement is static. Therefore, when polarity is reversed across the device, a hysteretic behavior may occur in p-q as well as e-f coordinates.

References

1 Paynter, H. M., *Analysis and Design of Engineering Systems*, The MIT Press, Cambridge, Mass., 1961.

2 Shearer, J. L., Murphy, A. T., and Richardson, H. H., *Introduction to Systems Dynamics*, Addison-Wesley, Reading, Mass., 1967.

3 Chua, L., "Memristor—The Missing Circuit Element," *IEEE Transactions on Circuit Theory*, Sept. 1971.

4 Purcell, E. M., *Electricity and Magnetism, The Berkeley Physics Course*, Vol. II, McGraw-Hill, New York, 1966.

5 Oster, G., and Auslander, D., "Topological Representations of Thermodynamic Systems," *Journal of the Franklin Institute*, Parts I, II, 1971.

Printed in U.S.A.